組合數學研討會
論文摘要

Workshop on Combinatorics
June 27-28, 1997

主辦單位：靜宜大學應用數學系
協辦單位：行政院國科會數學中心
時　　間：民國 86 年 6 月 27-28 日
地　　點：台中縣靜宜大學理學院

1997組合數學研討會

民國86年6月27-28日

靜宜大學理學院應用數學系

六月二十七日(星期五)

09:15-09:30 歡迎與感謝

第一節 主席：張鎮華教授(交通大學應數系)

09:30-10:00 葉鴻國 (交通大學)

Algorithmic Aspects of Majority Domination **(P.1)**

10:00-10:30 陳介宇 (交通大學)

Partition of Weighted Trees **(P.2)**

10:30-10:45 休息

第二節 主席：林 強教授(中央大學數學系)

10:45-11:15 徐泰煒 (中央大學)

The Decomposition of Complete Graphs, Complete Bipartite Graphs and Crowns **(P.3)**

11:15-11:45 李明茹 (中央大學)

The Double Star Decomposition of Complete Bipartite Graphs, Crowns and Complete Graphs **(P.5)**

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13:30-14:00 林俊沅 (交通大學)

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15:15-15:45 嚴志弘 (東海大學)

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   Optimum (t,n)-Families **(P.12)**

2. 李國偉(中央研究院數學研究所)

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六月二十八日(星期六)

第六節　主席：傅恆霖教授(交通大學應數系)

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09:30-10:00　方瑗 萲(交通大學)
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目　　錄

Algorithmic Aspects of Majority Domination

Student: Hong-Gwa Yeh Advisor: Dr. Gerard J. Chang

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ABSTRACT

This paper studies algorithmic aspects of majority domination, which is a variation of dominaton in graph theory. A majority dominating function of a graph $G = (V, E)$ is a function g from V to $\{1, -1\}$ such that $\sum_{v \in N[v]} g(v) \geq 1$ for at least half of the vertices $v \in V$. The majority domination problem is to find a majority dominating function g of a given graph $G = (V, E)$ such that $\sum_{v \in N[v]} g(v)$ is minimized. The concept of the majority dominatiom was introduced by Hedetniemi and studied by Broere et al., who gave the exact values for the majority domination numbers of the complete graphs, complete bipartite graphs, paths, and unions of two complete graphs. they also proved that the majority domonation problem is NP-complete for general graphs; and asked if the problem NP-complete for trees.

The main result of this paper is to give polynomial-time algorithms for the majority domination problem in trees, cographs, and k-trees with fixed k.

Partition of Weighted Trees

Student: Chieh-Yu Chen Advisor: Dr. Gerard J. Chang

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ABSTRACT

In models of manys real applications, we very often need to partition a tree into as many subtrees as possible and keep the weight of every subtree larger than a given constant. In this thesis, we design efficient algorithms for the above tree partition problem under different weight requirements.

The Decomposition of Complete Graphs, Complete Bipartite Graphs and Crowns

Student: Tay-Woei Shyu Advisor: Chiang Lin

Department of Mathematics

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Abstract

The problem of decomposing a graph into some subgraphs is an important subject of graph theory. There are many types of decomposition problems, such as clique decomposition, star decomposition, path decomposition, cycle decomposition, bipartite decomposition, complete bipartite decomposition, and so on. In this thesis we will study star decomposition, path decomposition and complete bipartite decomposition of some graphs.

There are five chapters in this thesis. In Chapter 1, some basic definitions and notations are introduced. In Chapter 2, we study the complete bipartite decomposition of the crown. The results can be applied to the directed complete bipartite decomposition of the complete symmetric directed graphs.

In Chapter 3, we study the star decomposition of complete graphs. We first give a necessary and sufficient condition for the star decomposition of the

complete graph. We then establish a multiple version of the Landau's Theorem which characterizes the outdegrees of the vertices of an oriented complete multigraphs, and use this to obtain a result concerning the decomposition of a complete multigraphs into multistars.

In Chapter 4, we study the star decomposition of crowns and complete bipartite graphs. We first establish a necessary and sufficient condition for the isomorphic star decomposition of the multiple crown. We then establish a sufficient condition for a complete bipartite multigraph to be decomposed into a given sequence of stars (not necessarily isomorphic), which implies a criterion for the isomorphic star decomposition of the complete bipartite multigraph.

In Chapter 5, we study the path decomposition. We establish a criterion for the isomorphic path decomposition of the crown, and that for the isomorphic antidirected path decomposition of the complete symmetric directed graph.

The Double Star Decomposition of Complete Bipartite Graphs, Crowns and Complete Graphs

Student: Ming-Ju Lee   Advisor: Chiang Lin

Department of Mathematics
National Central University

ABSTRACT

A double star $DS_{r,t}$ is the graph with vertex set $\{u_0, u_1, \ldots, u_r\} \bigcup \{v_0, v_1, \ldots, v_t\}$ and the edge set $\{u_0u_i : i = 1, 2, \ldots, r\} \bigcup \{v_0v_i : i = 1, 2, \ldots, t\} \bigcup \{u_0v_0\}$. A double star is a special case of a caterpillar. In this paper, we investigate the decomposition of complete bipartite graphs, crowns and complete graphs into double stars. The decomposition into caterpillars are also considered.

Fault Tolerant Token Ring Embedding in Double Loop Networks *

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ABSTRACT

A double loop network $G(n; s_1, s_2)$ is a digraph with n nodes $\{0, 1, \ldots, n-1\}$ and $2n$ links of the form $i \rightarrow i+s_1$ (**mod** n) and $i \rightarrow i+s_2$ (**mod** n). A double loop network $G(n; s_1, s_2)$ is *LFT* if there is a hamiltonian cycle in every $G(n; s_1, s_2) - e$ where e is any link in the network. Similarly, a double loop network $G(n; s_1, s_2)$ is *NFT* if there is a hamiltonian cycle in every $G(n; s_1, s_2) - v$ where v is a node in the network. In this paper, we present necessary and sufficient conditions for LFT and NFT double loop networks, respectively.

Thm: $G(n; s_1, s_2)$ is LFT iff at least one of the following condition hold
a) $\gcd(d, s_1) = 1$ and $\exists k$ $1 \le k < d$ s.t. $\gcd(ks_1 + (d-k)s_2/d, s/d) = 1$
b) $\gcd(n, s_i) = 1$, $\forall i = 1, 2$

*This work was supported in part by th National Science Council of the Republic of China under contract NSC86-2213-E009-020.

$s = |s_1 - s_2|$
$d = \gcd(n, s)$

Broadcasting in Communication Networks

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Abstract

In the broadcasting problem, exact one member of the network, called the broadcast originator (center), has a unique message which is to be disseminated to all other members by a call sequence over the communication network. In this thesis, we model a communication network by a graph $G = (V, E)$ and determine the minimum broadcast time in G. We prove that if G is a connected graph, then the minimum broadcast time in G is at least the radius of G; and if G is a connected graph in which there are two or more vertices that are farest to a center of the graph, then the minimum broadcast time is greater than the radius of G. We also prove that if G is a connected graph and $S \subseteq V(G)$ know the message by an optimal call sequence and there are at least two vertices that are farest to S, then the minimum broadcast time is greater than the maximum distance from a vertex to S plus the time needed for all vertices to know the message. We

use above results to obtain the broadcast times for Cartesian product and direct product of paths and cycles.

Distance-regular Graph with Classical Parameters

Student: Mine-Hway, Huang Advisor: Chih-Wen Weng

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ABSTRACT

Suppose $\Gamma = (X, R)$ is a distance-regular graph with classical parameters (d, b, α, β) We computer a general formula for the multiplicities of the adjacency matrix of Γ. Then we consider the special case $\alpha = \frac{b-1}{2}$ and $\beta = \frac{-b^d-1}{2}$. We find the formula does not suit the special case. We do a little change for this formula so that can suit this case.

Equitable Coloring of 3-partite Graphs

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Abstract

If the vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots,$ V_k such that each V_i is an independent set and $\big|\ |V_i| - |V_j|\ \big| \leq 1$ for all $i \neq j$, then G is said to be equitably k-colorable. In this thesis, we will study that the equitable Δ-coloring of bipartite graphs and 3-partite k-regular graphs.

The Total Star Number of Graphs

Student: Jyi-Shyang Leu Advisor: Dr. Yi-Wu Chang

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ABSTRACT

A multiple-star representation of a simple graph G assigns each vertex a union of stars in a host tree, such that vertices are adjacent if and only if their assigned sets intersect. The total star number $S(G)$ is the minimum of the total number of stars used in any such representation of G. We obtain the maximum value of $S(G)$ for m-edge connected graphs: $m+1$, n-vertex graphs: $[\frac{n^2+1}{4}]$, and n-vertex outer-planar graphs: $[\frac{3n}{2}-1]$.

Optimum (t, n)-Families

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May 22, 1997

Abstract

Let $F = (A_1, A_2, \cdots, A_n)$ be a family of subsets of a finite set X and $[n]$ be the index set $\{1, 2, \cdots, n\}$. An SDR (system of distinct representatives) for F is an n-tuple $(a_1, a_2, \cdots, a_n)$ such that $a_i \in A_i$ for all $i \in [n]$ and $a_i \neq a_j$ if $i \neq j$. A family $F = (A_i : i \in [n])$ is a (t, n)-family, if $|A(I)| \geq |I| + t$ for any nonempty subset $I \subset [n]$, where $A(I) = \cup_{i \in I} A_i$. A (t, n)-family F is minimum if the number of SDRs for F is the minimum of those of all (t, n)-families. J.G. Chang [Ch89] characterized the minimum (t, n)-families for $t = 0, 1, 2$ and conjectured that for $t \geq 3$, $F^* = (A_i^* : i \in [n])$ is the only minimum (t, n)-family where $A_i^* = \{i, n+1, n+2, \cdots, n+t\}, i \in [n]$. Leung and Wei [LeWe92] gave a proof via a comparison theorem for permanents but a flaw contained in their proof was found by Chang [Ch96].

This talk aims to discuss this conjecture and certain related problems.

References

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[LeWe92] J.Y.-T. Leung and W.-D. Wei, *A comparison theorem for permanents and a proof of a conjecture on (t, n)-families*, J. Combin. Theory, Ser. A, 61 (1992) 98-112.

A Problem on Sperner Theory

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ABSTRACT

A finite graded poset P is a finite partially ordered set with a rank function r such that $r(x) = 0$ for every minimal element x, and $r(x) = r(y) + 1$ whenever y covers x. We call $r(x)$ the rank of x. A set A of elements of P is an antichain in P if two arbitrary distinct elements of A are not related by the partial order of P. Let P_m denote the set of elements in P having rank m. P is said to be Sperner if $\max_m |P_m| = \max\{|A| : A \text{ is an antichain in } P\}$. A filter F in P is a subset of P such that, for any $a \in F$ and $b \in P$, $a \leq b$ implies $b \in F$. The principal filter $\langle a \rangle$ generated by a is defined to be the set $\{b : a \leq b\}$. A filter F is generated by $a_1, a_2, \ldots, a_k$, $k > 0$, if $F = \langle a_1 \rangle \cup \langle a_2 \rangle \cup \ldots \cup \langle a_k \rangle$. If we furthermore suppose that $a_1, a_2, \ldots, a_k$ are of a fixed rank, then the rank function r of P will induce a rank function r^* on F. That is $r^*(x) = r(x) - r(a_1)$ for all $x \in F$. F thus becomes a graded poset.

Let B_n denote the power set of $\{1, 2, ..., n\}$ ordered by set inclusion. It is well-known that B_n is a Sperner poset. In [3], I proposed the following conjecture and established the case for $t = 1$.

Conjecture. If F is a filter in B_n generated by a nonempty collection of t-subsets, then F is Sperner.

Griggs [1] first gave a collection of 4-subsets of B_6 which does not generate a Sperner filter. Then Zha [4] constructed counterexamples for any $t \geq 4$ and $n \geq 2t - 1$. Recently, Horrocks [2] has given an affirmative solution to the case for $t = 2$. Now the open problem is to settle the conjecture for the case $t = 3$.

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The Optimal Pebbling Number of Complete *m-ary* Tree *

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Abstract

In this paper, we find the optimal pebbling number of the complete *m-ary* trees.

key word. optimal pebbling, complete m-ary tree, integer linear programming

AMS(MOS) subject classification. 05C05

1 Introduction

Throughout this paper, a *configuration* of a graph G represents a *mapping* from $V(G)$ into the set of non-negative integers $N\bigcup\{0\}$. Suppose p pebbles are distributed onto the vertices of G, then we have the so-called distributing configuration(d. c.) δ where $\delta(v)$ is the number of pebbles distributed to $v \in V(G)$ and $\delta_H = \sum_{v\in V(H)} \delta(v)$ for each induced subgraph H of G. Note that now $\delta_G = p$.

A pebbling move consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a d. c. δ satisfies that we can move at least one pebble to each vertex v by applying pebbling moves repeatedly(if necessary), then δ is called a pebbling of G. The optimal pebbling number of G, $f'(G) = min\{\delta_G|\delta$ is a pebbling of $G\}$. And a d. c. δ is an *optimal pebbling* of G if δ is a pebbling of G such that $\delta_G = f'(G)$.

*Supported in part by the National Science Council of the Repubic of China(NSC-85-1221-M-009-010)

Note here that the *pebbling number* $f(G)$ of a graph G is defined as the *minimum* number of pebbles p such that no matter how we distribute these p pebbles onto G, then we have a *pebbling* of G.

A complete *m-ary* tree with heigth h denoted by T_h^m is an *m-ary* tree satisfying that v has m sons for each vertex v not in the h-th level. Let T be a complete *m-ary* tree with height h. If we distribute x_i pebbles on each vertex in the i-th level, $i = 0, 1, 2, ..., h$, then such a d. c. is denoted by $< x_0, x_1, ..., x_h >_T$. (If T is fixed then we omit T in notation.) We shall mainly use this type of d. c. to obtain the optimal pebbling number for complete *m-ary* tree, and the solution for $m = 2$ comes from solving a special integer linear programming. For clearness, we briefly introduce this term.

Give an $m \times n$ real matrix A and two vectors $\mathbf{b} \in R^m$ and $\mathbf{c} \in R^n$. Then the following optimization problem is an instance of integer linear programming(ILP).

$$min\ \mathbf{c}^T x$$

$$A\mathbf{x} \geq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

$$\mathbf{x} \in Z^n$$

The problem of *pebbling graph* was first proposed by M. Saks and J. Lagarias[1]. Since then some results have been obtained. First, F. R. K. Chung[1] showed that $f(Q_n) = 2^n$, and then D. Moews[2] proved that $f(G \times H) \leq f(G)f(H)$ for trees G and H. Later, L. Pachter et. al. [3] studied the *optimal pebbling* and the *optimal pebbling number* for a path was obtained.

Theorem 1.1. [3] *Let P be a path of order 3t+r, i.e., $|V(P)| = 3t + r$. Then $f'(P) = 2t + r$.*

Recently, Fu and Shiue[4] find $f'(T)$ for T a caterpillar. Furthermore, they show that $f'(G \times H) \leq f'(G)f'(H)$ for any graphs G and H.

In this paper, we shall focus on the study of the *optimal pebbling number* of a complete *m-ary* tree. We first obtain $f'(T)$ for a complete *m-ary* tree T with $m \geq 3$. And then, in Section 3, we show that the *optimal pebbling* problem of complete binary tree can be transformed to an instance of ILP and we find an efficient algorithm to find the optimal solution for the instance of ILP which is corresponding to the optimal pebbling number problem of complete binary tree.

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Optimal Packing and Covering of λK_v with Quadruples

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Abstract

In this paper, we study the optimal packing and covering of λK_v with quadruples(K_4). Mainly, we use minimum leave and minimum padding to describe a maximum packing and a minimum covering respectively. Then, the relationship between them will be clearly seen and also it's easier to apply these results to construct other designs, such as group divisible designs with block size 4 and two associate classes.

Partition Integral Set into Subsets with Prescribed Sums

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Abstract

A nonincreasing sequence of positive integers $\langle m_1, m_2, ..., m_k\rangle$ is said to be n-realizable if $I_n = \{1, 2, ..., n\}$ can be partitioned into k sets $S_1, S_2, ..., S_k$ such that for each $i \in \{1, 2, ..., k\}$, $\sum_{x \in S_i} x = m_i$. Clearly, if $\langle m_1, m_2, ..., m_k\rangle$ is n-realizable then $\sum_{i=1}^{k} m_i = \binom{n+1}{2}$.

In this thesis, we prove that a nonincreasing sequence of positive integers $\langle m_1, m_2, ..., m_k\rangle$ is n-realizable provided that $\sum_{i=1}^{k} m_i = \binom{n+1}{2}$ and $m_{k-1} \geq n$. This improves a known result obtained by Ma et.al. which assumes that $m_k \geq n$. Moreover, this is best possible in the consideration of part-size, for example, $\langle 7, 6, 1, 1\rangle$, $\langle 6, 6, 2, 1\rangle$ and $\langle 6, 5, 2, 2\rangle$ are not 5-realizable. As can be seen in each example the condition $m_{k-1} \geq n$ was dropped.

車輛途程問題在 T 型圖上的近似解法

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應用數學研究所計算機組

摘要

考慮使用一載車，將一組分散在某一區域中各個位置的物件，從它們的初始位置運送至各自的目的位置。如何尋找出將所有的物件分別運送到其目的位置，載車所須移動的最短距離，即爲車輛途程問題。對此問題根據不同的條件，車輪途程問題有多種不同的版本。在此篇論文中，我們所考慮所有的物件都是散落在 T 型圖上的四個點，而載車在同一時間可載運多於一個以上的物件，且載車只能沿著 T 型圖上的三個長度均爲 1 的邊移動。在物件搬運的過程中，物件可以被暫時放到其目的位置的點。我們提出解決此問題的一個近似演算法，而得到的近似解的結果最多爲最佳解的 1.5 倍。

Message Routing Algorithm on Directed Circulant Networks

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ABSTRACT

Communication between pairs of nodes is a fundamental problem in a distributed network. A routing scheme is to determine the path that a message from the source to the destination should take. In an optemal routing scheme, the path taken by any message must be a shortest path. The traditional way is to give each node a routing table of size $O(n)$, but it is not a suitable solution for large networks. There are some improved methods by using compact tables instead of complete tables such as interval routing.

A directed circulant network $G(n; s_1, s_2)$ is a directed graph with vertices $\{0, 1, 2, \ldots, n-1\}$, and $2n$ links of the form $i \rightarrow i + s_1(modn)$ and $i \rightarrow i + s_2(modn)$. We present an optimal message routing algorithm for directed circulant networks, and that for any message from node i to node j, each intermediate node k needs only constant time in determining the shortest route.

一些關於排列組合的演算法

(Some Algorithms about Permutations and Combinations)

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摘要

在已知的排列組合運算中，雖然已知的公式已經不少，但對於現實生活上所遇到的，往往不只是要求得到***"總共有多少個"***，最重要的會是在於***"到底有哪些個！！"***。在本篇之中，將利用電腦的輔助，將您所想要的結果一一列出來，您的問題不再是只能得到一個空洞的***"數字解"***，而是能完完全全地了解整個狀況，給您對於排列組合問題一種新的感受！

除此之外，對於排列組合中仍有許多不容易處理的問題，至今仍沒有一個簡單的公式解的，在本篇之中，雖然也一樣沒法告訴您它的公式解是什麼，但透過電腦的幫助，至少能在很短的時間之內，算出您想要的結果；且除了能夠將結果一一呈現列印在您的電腦螢幕上之外，更能在不需要浪費記憶體的情況之下，就可以把結果都保存下來。讓您能夠解決"總共有多少個"的問題，也同時能讓您知道"到底有哪些個"！

Linear arboricities of complete r-partite graphs

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Abstract

A linear forest of a graph G is a subgraph of G whose connected components are paths. The linear arboricity of G, denoted by $\mathrm{la}(G)$, is the minimum number of linear forests needed to partition the edge set $E(G)$ of G. In this paper, we study linear arboricities of complete r-partite graphs.

Vertex Ranking of Graphs

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Jun 1997

Abstract

A vertex ranking of a graph G is a function ρ from $V(G)$ to $\{1, 2, \cdots\}$ such that for every pair of distinct vertices x and y and for every (x, y)-path P, if $\rho(x) = \rho(y)$ then there exists an intermediate vertex z of P such that $\rho(x) < \rho(z)$. If $\rho(x) = i$, then we say that x has a rank i. The vertex ranking problem is to find a vertex ranking of a given graph using the minimum number of ranks. We define such a function an optimal vertex ranking. The rank number $r(G)$ is the number of ranks used in any optimal vertex ranking of G. In this paper, we find rank numbers of some graphs and design an efficient algorithm for the vertex ranking problem in block graphs.

Two-Dimentional T-Coloring Problems on Graphs

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Abstract

This article proposes *two-dimensional T-colorings* on simple graphs. The ordinary T-coloring will be the one-dimensional version of our T-colorings. Let T be a set of nonnegative integers that containing 0. Denote $Z_0^2 = \{(x_1, x_2) : x_i \in Z_0 \ , i = 1, 2\}$, where $Z_0 = \{0, 1, 2, \cdots\}$. A *2-dimensional infinite-norm*, $\|\cdot\|_\infty$, is a function from Z_0^2 to Z_0 that $\|X\|_\infty = \max\{|x_1|, |x_2|\}$ for $X = (x_1, x_2,)$. A *2-dimensional one-norm*, $\|\cdot\|_1$, is a function from Z_0^2 to Z_0 that $\|X\|_1 = |x_1| + |x_2|$ for $X = (x_1, x_2,)$ in Z_0^2. With these norms, we define a *2-dimensional T-coloring with respect to p-norm* ($p = 1$ or ∞), f, on a graph $G = (V, E)$ as follows:

$f : V \to Z_0^2$ such that $\|f(u) - f(v)\|_p \notin T$, for all $\{u, v\} \in E$. They are denoted by *$T(2,p)$-colorings* . Note that both T-colorings for $d = 1$ are same as usual T-colorings.

The *edge span* of a $T(2,p)$-coloring f on a graph G, denoted by $esp_T(f; 2, p)$, is the maximum value of $\{\|f(u) - f(v)\|_p : \{u, v\} \in E(G)\}$. The *edge span of G* with respect to $T(2,p)$-coloring is the smallest number m such that there is a $T(2,p)$-coloring , f with $esp_T(f; 2, p) = m$. It is denoted by $esp_T(G; 2, p)$.

As previous problem on T-coloring, we are interested in studying the edge spans for both colorings with different T-sets.

The L(2,1)-edge-span of a Graph

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Abstract

Let $G = (V, E)$ be a simple graph. A function f is called an $L(2,1)$-*labelling* of G, if $f : V \to \{0, 1, 2, \cdots\}$ such that (1) $|f(u) - f(v)| \geq 2$ whenever $\{u, v\} \in E$ and (2) $|f(u) - f(v)| \geq 1$ whenever the distance between u and v is two. The minimum number of m such that there is an $L(2,1)$-labelling on G with maximum value m is called the $L(2,1)$-*number* of G and is denoted by $\lambda(G)$. Since this labelling been proposed, most of the works are concentrated on studying λ. However this article will propose another parameter regarding the $L(2,1)$-labelling.

Let us define $\beta(G)$ to be the minimum number of m such that there is an $L(2,1)$-labelling, f, on G with $max\{|f(u) - f(v)| : \{u, v\} \in E(G)\} = m$. This number $\beta(G)$ is called the $L(2,1)$-*edge-span* of G. It is obvious that $\lambda(G) \geq \beta(G)$. How small can β be and when the equality will hold? We will study the behavior of β on defferent classes of graphs and try to answer the questions above.

Open Problem 9501 (Taiwan Combinatorics Group)
Date: February 1995 (award: NT$200+NT$300)

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A system of distinct representatives (SDR) of a family $F = (A_1, A_2, \ldots, A_n)$ is a sequence $(a_1, a_2, \ldots, a_n)$ of n distinct element with a_i in A_i for $1 \leq i \leq n$. A theorem of P. Hall [3] says that a family has a SDR if and only if the union of k sets of this family contains at least k elements. Denotes by $N(F)$ the number of SDR of family F; two SDR are considered distinct if they are different in at least two components. Several quantative refinements of P. Hall's theorem were given by M. Hall [2], Rado [6], and Mirsky [5]. Their results are all under the assumption of P. Hall's condition plus some extra conditions on the cardinalities of A_i's.

The author [1] considered an extension of P. Hall's condition as follows. Let t be a non-negative integer. A (t,n)-family is a family $F = (A_1, A_2, \ldots, A_n)$ such that the union of any $k \geq 1$ sets of the family contains at least $k+t$ elements. Denotes by $M(t,n)$ the minimum of $N(F)$ where F runs over all (t,n)-families. P. Hall's theorem says that $M(0,n) \geq 1$; in fact, $M(0,n) = 1$.

Consider the (t,n)-family $F^* = (A_1^*, A_2^*, \ldots, A_n^*)$, where $A_i^* = \{i, n+1, n+2, \ldots, n+t\}$ for $1 \leq i \leq n$. It is easy to see that

$$N(F^*) = U(t,n) \equiv \sum_{j=0}^{\min(t,n)} j! \binom{t}{j} \binom{n}{j}.$$

So $U(t,n)$ is an upper bound for $M(t,n)$. [1] proved that $M(1,n) = U(1,n) = n+1$ and $M(2,n) = U(2,n) = n^2+n+1$. For $0 \leq t \leq 2$, all (t,n)-families F with $N(F) = M(t,n)$ were also determined. In particular, F^* as above is the only $(2,n)$-family F with $N(F) = M(t,n)$.

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* [1] conjectured that $M(t,n) = U(t,n)$ and F^* is the only *
* (t,n)-families F with $N(F) = M(t,n)$ for all $t \geq 3$. *

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Let $B = (b_{i,j})$ be an n by m matrix over a ring R. The permanent of B is defined as $per(B) = \sum_{j_1 j_2 \ldots j_n} \prod_{1 \leq i \leq n} b_{i,j_i}$, where $j_1 j_2 \ldots j_n$ is an n-permutation of $\{1, 2, \ldots, m\}$. When $n > m$, $per(B) = 0$. For a family $F = (A_1, A_2, \ldots, A_n)$, where each A_i is a subset of $S = \{x_1, x_2, \ldots, x_m\}$, the incidence matrix of F is the $n \times m$

matrix $A = (a_{i,j})$ defined by $a_{i,j} = 1$ if $x_j \in A_i$ and $a_{i,j} = 0$ otherwise. It is easy to see that $N(F) = per(A)$.

Leung and Wei [4] gave a proof of the above conjecture by means of a comparison theorem for permanents. However, their proof has a flaw at the bottom of page 109. For the (t, n)-matrix A, the new matrix A' obtained from A by applying Theorem 1 of [4] is in not necessary a (t, n)-family. The following is an example of a $(3,3)$-matrix A for which A' is not a $(3,3)$-matrix when use $p = 2$ and $q = 3$ as in Theorem 1:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad A' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A more convincing proof of the conjecture is desirable.

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Problems from the Thirteenth British Combinatorial Conference

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Most of these problems were presented at the Problem Session held at the conference. However, I have also included some problems from participants who were unable to attend the session, or remembered afterwards problems they should have presented. The originator of the problem is given, where known and different from the proposer. If no originator is given, either it is the proposer's problem, or my ignorance is to blame.

Problems are grouped roughly by subject matter.

The addresses of presenters are given at the end. Communication is encouraged; indeed, one presenter conjectured that the answer to his question is known!

1. Stable partitions of graphs

A partition of the vertex set V of a (simple) graph G into two non-empty parts V_1 and V_2 is called *stable* if, for $i = 1, 2$, and all vertices $x \in V_i$,

$$\deg_{G|V_i}(x) > \frac{1}{2}\deg_G(x).$$

A graph with a stable partition is also called *stable*. The complete graph is not stable, but every 3-regular graph other than K_4 is stable.

What is the number of stable graphs? In particular, is the proportion of graphs on n vertices which are stable $o(1)$?

Proposer: Z. Füredi.

2. Ramsey perfect graphs

Is it true that, for every prfect graph G, there exists a perfect graph H such that, for every partition $E(H) = E_1 \cup E_2$, there is an induced subgraph G' of H such that G' is isomorphic to G and $E(G') \subseteq E_i$ for some i ($i = 1$ or 2)?

We expect a negative answer to this question, despite the fact that the analogous question for partitions of vertices has an affirmative answer (using the Lovász Multiplication Lemma).

Proposer: J. Nešetřil.

3. Reconstructing spanning trees

Is the number of spanning trees with exactly d automorphisms reconstructible from the deck of vertex-deleted subgraphs of a graph, for each d?

Proposer: C. Wakelin. *Originator*: W. L. Kocay.

4. 1 time 2-tough implies 2 times 1-tough?

Prove that the edge set of a 2-tough graph G can be partitioned into two sets E_1 and E_2 such that each of E_1 and E_2 induces a 1-tough spanning subgraph of G.

If this conjecture is true, then a 4-regular 2-tough graph would consist of two edge-disjoint Hamiltonian cycles. One could accordingly extend Thomassen's conjecture to the assertion that any 2-tough graph contains two edge-disjoint Hamiltonian cycles.

(A graph is t-tough if, for any vertex cutset S, $G - S$ has at most $|S|/t$ components. Chvátal conjectured that, for $t > 3/2$, a t-tough graph is Hamiltonian. This was refuted by Thomassen, who conjectured instead that a 2-tough graph is Hamiltonian.)

Proposer: C. Hoede.

5. Path-tough graphs

Let G be a simple graph on n vertices, where $n \geq 3$, which is *path-tough* (that is, $G - v$ has a Hamiltonian path for any vertex v.) Suppose that $d(u) + d(v) + d(w) \geq n - 2$ for any three independent vertices u, v, w. Prove that either G is Hamiltonian, or G is isomorphic to the Petersen graph.

Proposer: I. Schiermeyer.

6. Hamilton cycles and a bit more

Prove that any graph of order n and minimum degree at least $\frac{1}{2}(n+1)$ contains a subgraph which is a Hamiltonian cycle together with a longest diagonal.

Proposer: R. Häggkvist.

7. The second longest cycle

Give lower bounds for the length of the second longest cycle in a Hamiltonian 3-regular graph. (The best bound currently is $n - 4\sqrt{n}$.) More generally, the same problem with "minimum degree 3" in place of "3-regular".

Proposer: R. Häggkvist.

8. 4-chromatic covering graphs

What is the smallest 4-chromatic covering graph? (A graph is a *covering graph* if its edges can be oriented in such a way that it becomes a Hasse diagram of a poset.) The answer lies between 12 and 14 inclusive.

Proposer: D. Youngs.

9. Special Hamiltonian paths

Consider the diagram of the poset of all subsets of $\{1, 2, \ldots, n\}$. As a graph, this is the n-cube. Does the graph have a Hamiltonian path $A_1, A_2, \ldots, A_{2^n}$ starting at $A_1 = \emptyset$ and having the following property?

- *If, at step i, you visit a set A_i, then you must previously have visited all subsets of A_i with at most one exception. If there is an exception, you must visit it next (that is, it is A_{i+1}).*

  For example, with $n = 4$,

$$\emptyset, 1, 12, 2, 23, 3, 34, 4, 24, 124, 14, 134, 13, 123, 1234, 234$$

is such a path.

Proposer: T. Trotter. *Originators*: T. Trotter & S. Felsner.

10. Interval-regular graphs

Let G be a connected simple graph. The *interval* $I(u, v)$ between vertices u and v is the set of vertices which lie on some shortest path from u to v. The graph is called *interval-regular* if, for any two vertices u and v, we have

$$|N(u) \cap I(u, v)| = d(u, v),$$

where $N(u)$ is the set of neighbours of u.

Examples of interval-regular graphs include hypercubes, and the 2-cube and 3-cube with added edges joining all vertices in $N(u)$ for some vertex u. The class of interval-regular graphs is closed under taking Cartesian products. See Mulder, *Math. Centre Tracts* **132** (1980), or *Discrete Math.* **41** (1983), 253–269.

Conjecture: Let G be interval-regular. Then

$$x, y \in I(u, v) \Rightarrow I(x, y) \subseteq I(u, v)$$

for any vertices u and v.

Proposer: H. M. Mulder.

11. Permutations of a cube

Let C be the n-cube graph, d the graph metric. Is any permutation of the vertices of C the composite of at most n permutations s_i, each satisfying

$$d(x, s_i(x)) \leq 1$$

for all $x \in C$?

Editor's note: More generally, which graphs of diameter n have this property?

Proposer: M. K. Siu

12. Permutations of projective space

For which n and q does there exist a permutation π of the point set of $PG(n,q)$ with the property that, for any hyperplane H, there exists a hyperplane H' with $\pi(H) \cap H' = \emptyset$? It is known that the answer is "yes" for $n = 2$ (asymptotically, almost all permutations have this property), and "no" for $n > q$ (by a short argument due to A. Blokhuis).

Proposer: P. J. Cameron. *Originator*: A. Gyárfás.

13. q-polynomials

A polynomial $c(z) = \sum c_i z^i$ over $GF(q)$ is said to be a *q-polynomial* if $c_i \neq 0$ only if i is a power of q. (See Lidl & Niederreiter, *Finite Fields*, Cambridge Univ. Press.) Let $m = 2^d - 1$ be a Mersenne prime. Does there exist a 2-polynomial

$$c(z) = \sum_{i=0}^{d} c_i z^{2^i}$$

of degree 2^d such that $c(z)/z$ is irreducible over $GF(2)$? (The answer is "yes" for the first four Mersenne primes, viz. 3, 7, 31, 127.)

Proposer: A. Bonisoli.

14. Cyclic shifts of binary words

For odd n, let W_n be the space of binary words of length n and even weight. Let $f(n)$ be the maximum codimension of a subspace of W_n such that the union of all cyclic shifts of U is equal to W_n. It is known that $f(n) \geq 2$ for $n > 3$, but little more is known. Does $f(n) \to \infty$ as $n \to \infty$? Or is $f(n) = 2$ for infinitely many n?

Proposer: P. J. Cameron.

15. A generalisation of arcs

Let $(V, \mathcal{B})$ be an affine plane of odd order q. Given a function $w : V \to \mathbb{Z}$ such that

$$\sum_{p \in L} w(p) \leq 2$$

for all $L \in \mathcal{B}$, set

$$\text{mass}(w) = \sum_{p \in V} w(p).$$

Then $\text{mass}(w) \leq q + 2$ holds. The value $q + 1$ can be attained (by the characteristic function of a conic). Prove that necessarily $\text{mass}(w) \leq q + 1$.

Proposer: J. Bierbrauer.

16. Perfect Steiner triple systems

Let $(V, \mathcal{B})$ be a Steiner triple system of order v (an STS(v)). For distinct points x, y, contained in the block $\{x, y, z\}$, the *interlacing graph* G_{xy} is the 2-regular graph on the vertex set $V \setminus \{x, y, z\}$, in which a and b are adjacent whenever $\{a, b, x\}$ or $\{a, b, y\} \in \mathcal{B}$. The STS is called *perfect* if G_{xy} is a Hamiltonian cycle for all x, y.

Problem: Find more perfect STS. Are there infinitely many? (Only four are known, with orders 7, 9, 25, and 33. There is none of order 13 or 15. All known examples have point-transitive automorphism groups.)

Proposer: A. Rosa. *Originators*: C. J. Colbourn & A. Rosa.

17. A generalisation of affine designs

An *affine design* is a 2-design in which any two non-parallel blocks meet in a constant number y of points. What happens if we replace "non-parallel" by "non-disjoint" in this definition? In addition to the affine designs, there are resolvable designs with $\lambda = 1$ (and $y = 1$), for example Kirkman systems. No others are known; the first unknown case is a resolvable 2-(70, 10, 6) design with $y = 2$.

Proposer: P. J. Cameron.

18. Blocking-set-free configurations

A configuration n_3 has n points and n lines, with three points on each line and three lines through each point, such that two points lie on at most one line. A *blocking set* is a set of points meeting every line but containing none. For which $n \geq 7$ does there exist a connected n_3 configuration with no blocking sets?

It is known that such configurations exist for all but finitely many values of n. They do not exist for the values 8–12 or 14. The values in doubt are 15–18, 20, 23, 24, 26, 29, 30, 32, 38, 44. See J. W. DiPaola and H. Gropp, *Mitt. Math. Sem. Gießen* (to appear).

Proposer: H. Gropp. *Originators*: J. W. DiPaola & H. Gropp.

19. Permutations with few distances

For fixed s, or for $s < n/2$, the best known upper and lower bounds for the maximum number of permutations of an n-set with s differences are both roughly $(cn/s)^{2s}$ (with different values of c. Find the correct value.

Proposer: P. J. Cameron.

20. Partial transversals of Latin rectangles

Let R be an $n \times 2n$ Latin rectangle on $2n$ symbols. A partial transversal T of size s of R is a collection of s cells, no two in the same row or column, and no two containing the same symbol. Is it true that R can be expressed as the union of $2n$ partial transversals of size n?

An equivalent formulation: Call two $n \times 2n$ Latin rectangles R, S on the same set of symbols *orthogonal* if the pairs (r_{ij}, s_{ij}), for $i = 1, \ldots, n$, $j = 1, \ldots, 2n$, are all distinct. Does every $n \times 2n$ Latin rectangle have an orthogonal mate?

Proposer: A. J. W. Hilton.

21. Semi-Latin squares

A *semi-Latin square* is an $n \times n$ array with k symbols (chosen from an alphabet of size nk) in each cell, so that each symbol occurs once in each row or column. We impose the further property

- *No two symbols occur together in a cell more than once.*

Such a structure clearly exists if there are k mutually orthogonal Latin squares of order n, on distinct sets of symbols.

Problem: Find constructions for values (n, k) for which a set of k m.o.l.s. of order n does not exist (or is unknown). (Examples are known for $(n, k) = (6, 2)$ or $(6, 3)$. What about $(6, 4)$? $(10, 3)$?)

Proposer: R. A. Bailey.

22. Tiling the square

(a) What is the least *odd* number of congruent non-rectangular tiles needed to tile a square?

(b) Is there such a tiling in which the tiles are not polyominoes?

The best tiling known to the proposer uses 25 copies of the polyomino with two rows containing 6 and 3 squares, aligned at one end.

Proposer: D. Youngs.

23. Covering the square

(a) Prove that, for any partition of the plane into sets (or regions) of diameter 1, the density must be at least $8/3\sqrt{3}$.

(b) A finite variation. What is the largest square which can be partitioned into n sets of diameter (at most) 1? The answer is known for $n \leq 5$. In general, we would expect a hexagonal honeycomb with some distortion at the edges.

An equivalent formulation asks for the chromatic number of the graph whose vertices are the points of the unit square, two points adjacent if their distance exceeds d.

Proposer: F. Barnes.

24. Sum-free sets containing 2

What is the probability that, in a random sum-free set S of natural numbers, 2 is the only even number in S? (Is it zero or not?)

(The probability measure is defined by the following rule. Consider the natural numbers in their usual order. If n is the sum of two numbers in S, then $n \notin S$; otherwise, decide on the toss of a fair coin. It is known that the probability that S contains no even numbers is non-zero, but the present problem seems a bit more delicate.)

Proposer: N. J. Calkin. *Originator*: P. J. Cameron.

25. Partitions of intersecting families

Let $\mathcal{F}$ be an intersecting family of k-subsets of the n-element set V, that is, $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. Let $p(\mathcal{F})$ be the minimum p for which one can find p pairs (2-subsets) $P_1, \ldots, P_p$ of V such that every member of $\mathcal{F}$ contains some P_i. Now let $f(n,k)$ be the maximum of $p(\mathcal{F})$, over all such intersecting families. Is it true that $f(n,k) \leq n$ for all n?

If true, this would imply a strengthened form of the case $t = 1$ of Larman's conjecture, since

$$\mathcal{F}_i = \{F \in \mathcal{F} : P_i \subseteq F, P_j \not\subseteq F \text{ for } j < i,$$

for $i = 1, \ldots, n$, is a decomposition of $\mathcal{F}$ into 2-intersecting families.

(*Larman's conjecture* (D. G. Larman, *Ann. Discrete Math.* 20 (1984)), asserts that if $\mathcal{F}$ is a t-intersecting family of k-subsets of the n-set V, that is, $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$, then $\mathcal{F}$ can be decomposed into n subfamilies each of which is $(t+1)$-intersecting.)

Editor's Note: A similar question can be asked for arbitrary families of sets (not all of the same size).

Proposer: Z. Füredi. *Originators*: N. Alon, P. Seymour & Z. Füredi.

26. Some families of sets

What can be said about families $\mathcal{F}$ of subsets of an n-set V satisfying

(i) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \not\subset F_2$;

(ii) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \neq \emptyset$;

(iii) $(\forall x \in V)(\exists F_1, F_2 \in \mathcal{F})\ F_1 \cap F_2 = \{x\}$.

Proposer: N. J. Calkin.

27. Some problems on perfect groups

(a) Prove the Feit–Thompson theorem by elementary means. (Remark: either of the following equivalents of the Feit–Thompson theorem may be more convenient:

- A finite perfect group can be generated by a self-inverse conjugacy class of elements of odd order (Heineken, unpublished);
- If G has odd order n, then some element of G is not the product of n distinct factors (Dénes and Hermann, *Ann. Discrete Math.* 15 (1982), 105–109).)

(b) Show that a perfect group can be generated by an involution and an element of odd order which is conjugate to its inverse.

(c) Characterise finite groups in which every element is a commutator. In particular, show that every non-abelian simple group has this property (Ore's conjecture).

(d) Characterise finite groups G in which every element of the derived group is the product of k commutators. (This condition can be expressed in terms of the notion of k-conjugacy (Yff, *Proc. Edin. Math. Soc.* 14 (1964), 1–4).)

Proposer: P. Yff. *Originators*: J. Dénes & P. Yff.

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Version of 3 September 1991

Problems from the Fourteenth British Combinatorial Conference

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Version of 6 January 1994

Most of these problems were presented at the Problem Session held at the conference. However, I have also included a couple of problems raised immediately after the conference. The originator of the problem is given, where known and different from the proposer. If no originator is given, either it is the proposer's problem, or my ignorance is to blame. Problems are grouped roughly by subject matter. The addresses of presenters are given at the end.

Two of the problems have now been solved, those of Douglas West on the bandwidth of a triangular lattice graph (solved by Rob Hochberg, Colin McDiarmid and Mike Saks), and Peter Cameron on a bijection between permutations with even and odd cycles (solved by Simon Norton and independently by Richard Lewis). Papers containing these solutions have been submitted for publication in the Conference proceedings; the problems in question have been left in the list for reference.

1. Total colourings of hypergraphs

A *total colouring* of a hypergraph is a colouring of vertices and edges such that

(a) the restrictions to vertices and edges are strong colourings;

(b) an incident vertex and edge have different colours.

The *total chromatic number* is the least number of colours required for a total colouring.

Conjecture. If $H = (V, \mathcal{E})$ is a linear hypergraph (two vertices on at most one edge) with total chromatic number $\chi_T(H)$, then

$$\chi_T(H) \leq \max_{x \in V} \left| \bigcup_{\substack{E \in \mathcal{E} \\ x \in E}} E \right| + 1.$$

Proposer: P. Cowling.

2. Critical K_l-free graphs

It is known that k-chromatic critical graphs on n vertices have at least $\left(\frac{k-1}{2}\right) n$ edges. Gallai showed that a better lower bound holds for graphs containing no K_k. Can this bound be further improved for graphs containing no K_l, for fixed l with $3 \leq l \leq k$?

For example, with $k = 10$, $l = 9$, can the bound $4.5n$ be improved to $5n - 10$?

Proposer: J. Schönheim.

Solved..

3. Bandwidth of a graph

The *bandwidth* of an n-vertex graph G is

$$\min_f \max_{x\sim y} |f(x) - f(y)|,$$

where the minimum is over all bijections from the vertex set to $\{1, \ldots, n\}$.

What is the bandwidth of the "triangular lattice" graph whose vertices are all triples of non-negative integers with sum l, vertices (x, y, z) and (x', y', z') being adjacent whenever $|x - x'| + |y - y'| + |z - z'| = 2$? (A lower bound of $l/2$ is known, and an upper bound of $l + 1$ is obtained by numbering the vertices in layers.)

Proposer: D. B. West.

4. How small is Tutte's wheel?

W. T. Tutte proved that any 3-connected graph can be obtained from a wheel by repeatedly adding an edge or splitting the central vertex (keeping the minimum degree at least 3).

Conjecture. Any 3-connected cubic graph on n vertices may be obtained by this procedure from a wheel on k vertices, where $k \geq cn$ (for some absolute constant c).

Proposer: A. Shastri.

5. Characteristic polynomials of graphs

How many distinct characteristic polynomials of (adjacency matrices of) n-vertex graphs are there?

The proposer conjectures that a typical n-vertex graph has n^2 cospectral mates, so that the answer to the problem is $O(2^{n(n-1)/2}/n^2 n!)$.

Proposer: R. Häggkvist.

6. Cliques and cocliques in Cayley graphs

Conjecture. There is a constant c such that, for every finite group G of order $n > 1$, there is a symmetric (i.e., inverse-closed) generating set S for G such that the Cayley graph $\Gamma(G, S)$ has neither a clique nor an independent set of size $c \log n$.

This is not known for any infinite sequence of finite groups; but it is true with $\log^2 n$ replacing $\log n$.

Proposer: N. Alon.

7. Local structure in 2-transitive graphs

Problem. Determine the vertex and edge stabilizers in all locally finite 2-transitive graphs in which $G_1(x) = 1$.

(A graph is 2-transitive if it admits a group G acting transitively on 2-arcs. The condition $G_1(x) = 1$ means that a vertex stabilizer acts faithfully (and 2-transitively) on its neighbours. The answer to this problem would be a list of pairs (H, t), where H is a finite 2-transitive group (the vertex-stabiliser $G(x)$) and t an outer automorphism of order 2 of the stabilizer H_y (so that $H_y\langle t\rangle = G(e)$, where $e = \{x, y\}$), along with the trivial possibility that $G(e) = H_y \times 2$.)

Proposer: A. A. Ivanov.

8. The rows of a Latin square

It is known that, for almost all Latin squares of order n (i.e., a proportion tending to 1 as $n \to \infty$), the rows of the square (regarded as permutations) generate S_n or A_n. Is this statement still true if the squares are normalized so that the first row is the identity permutation?

A subsidiary problem: Is it true that we almost always obtain the symmetric, rather than the alternating, group?

Proposer: P. J. Cameron.

Editor's note: Roland Häggkvist and Jeanette Janssen have given an affirmative solution to the subsidiary problem: they show that the proportion of Latin squares in which all rows are even permutations is exponentially small. They suggest a strengthening of this problem. Is it true that the distribution of the number of rows of a random Latin square which are odd permutations is "approximately" binomial $B(n, \frac{1}{2})$?

The following problem is also relevant. Let $M(n)$ and $m(n)$ denote the maximum and minimum numbers of extensions of a $2 \times n$ Latin rectangle to an $n \times n$ Latin square. Find a good upper bound for $M(n)/m(n)$.

More generally, how do you choose a random Latin square of order n?

9. A bijective proof of the Dyson conjecture

Let $R(r, m, n)$ denote the set of partitions of n whose rank is congruent to r modulo m, where the *rank* of a partition is the largest part minus the number of parts. Freeman Dyson conjectured, and Atkin and Swinnerton-Dyer proved, that

$$|R(0, 5, 5n + 4)| = |R(1, 5, 5n + 4)| = \ldots = |R(4, 5, 5n + 4)|.$$

The problem is to find a bijective proof.

Proposer: R. Lewis. *Originator*: Freeman Dyson, George Andrews.

10. Even and odd permutations

For even n, the number of permutations of $\{1, \ldots, n\}$ with all cycles of even length is equal to the number of permutations with all cycles of odd length. Find a bijective proof of this fact.

Proposer: P. J. Cameron.

11. How many sum-free sets?

Let $s(n)$ be the number of sum-free subsets of $\{1, \ldots, n\}$ (i.e., containing no solution to $x + y = z$). Show that there exist constants c_o and c_e such that $s(n)/2^{n/2} \to c_o$ or c_e as $n \to \infty$ through odd or even values respectively.

It is known only that $s(n) = 2^{(\frac{1}{2}+o(1))n}$.

Proposer: P. J. Cameron. *Originator*: P. J. Cameron & P. Erdős.

12. Non-crossing queens

What is the maximum number of non-crossing n-queens? It is known that the maximum is n if n is prime.

Proposer: G. B. Khosrovshahi.

13. Block-transitive designs

A t-(v, k, λ) design has v points and a collection of blocks of size k, any t points lying in exactly λ blocks. Terms such as "block-transitive" apply to the action of the automorphism group.

(a) Show that there is no block-transitive 6-design.

(b) Show that a block-transitive, point-imprimitive 3-design satisfies $v \leq \binom{k}{2} + 1$.

(c) Is there a block-transitive 2-$(\infty, 4, 1)$ design which is not point-transitive?

Proposer: P. J. Cameron. *Originator*: P. J. Cameron and C. E. Praeger.

14. Blocking-set-free configurations

A configuration n_3 has n points and n lines, with three points on each line and three lines through each point, such that two points lie on at most one line. A *blocking set* is a set of points meeting every line but containing none. For which $n \geq 7$ does there exist a connected n_3 configuration with no blocking sets?

It is known that such configurations exist for all but finitely many values of n. They do not exist for the values 8–12 or 14. The values in doubt are 15–18, 20, 23, 24, 26. (The value 15 may now be settled). See J. W. DiPaola and H. Gropp, *Mitt. Math. Sem. Gießen* (to appear).

Proposer: H. Gropp. *Originators*: J. W. DiPaola & H. Gropp.

Editor's note: This problem is repeated in updated form from the Problem Session at the 13th BCC.

15. Arranging rows and columns

A matrix of zeros and ones is said to be "zero-partitionable" if its rows and columns can be permuted independently so that the zeros of the resulting matrix can be labeled R or C such that

- every position to the right of an R is a 0 labeled R, and
- every position below a C is a 0 labeled C.

What is the complexity of recognizing zero-partitionable matrices?

(This is equivalent to recognition of interval digraphs. If a 0 is allowed to receive both R and C, this becomes recognition of digraphs with Ferrers dimension 2, which runs in polynomial time.)

Proposer: D. B. West

16. Piles of counters

I learned this problem just after the conference. Suppose that we have n counters divided into r non-empty piles, of sizes $n_1, \ldots, n_r$. A *move* consists in replacing two piles of sizes n_i and n_j by a single pile of size $n_i + n_j$; its *payoff* is 2 if $n_i = n_j$, 1 otherwise. After $r - 1$ moves, we have a single pile of size n.

(a) Is there a simple formula for the maximum total payoff which can be achieved?

(b) If not, what is the complexity of calculating the maximum payoff?

This problem has a group-theoretic origin. For the case where $n_i = 1$ for $i = 1, \ldots, r = n$, the maximum is $2n - b(n) - 1$, where $b(n)$ is the number of ones in the base 2 expansion of n.

Proposer: P. J. Cameron. *Originator*: R. Solomon.

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Problems from the 15th British Combinatorial Conference

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(First official draft, September 1995.)

Most of these problems were presented at the problem session at the conference on 7 July 1995. A few others, given to me before or after the session, have been added. I have used an approximation to the *Discrete Mathematics* format for the problems this year. In particular, the problem number given will change to that in the *Discrete Mathematics* sequence when the problems are published. The BCC numbers will remain the same for reference purposes.

Problem 1 (BCC15.1). A generalization of Hadwiger's conjecture. Posed by Ding, Oporowski, Sanders, and Vertigan.

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A *vertex partition* of G is a set $\{A_1, \ldots, A_k\}$ of induced subgraphs such that $V(G)$ is the disjoint union $V(A_1) \cup \ldots \cup V(A_k)$.

Conjecture. Every graph with no K_n minor has a vertex partition into $n-m+1$ graphs with no K_m minor.

For $m = 2$, this is Hadwiger's conjecture. It is known to be true for $n \leq 5$ (Wagner [3]; Ding, Oporowski, Sanders, Vertigan); for $n = 6, m = 2$ (Robertson, Seymour, Thomas [2]), and for $6 \leq n \leq 8$, $m = 3$ (Jørgensen [1]).

. L. K. Jørgensen, Contractions to K_8, *J. Graph Theory* **18** (1994), 431–448.
. N. Robertson, P. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.
. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.

Problem 2 (BCC15.2). Uniquely total colourable graphs. Posed by M. Behzad and E. S. Mahmoodian.

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A *total colouring* of a graph is a colouring of the vertices and edges in such a way that no two adjacent or incident elements have the same colour.

Problem. Show that, apart from empty graphs, paths, and cycles C_{3k}, there is no graph which has a unique total colouring (in the minimum number of colours).

A prize of 500000 Iranian rials is offered for this problem.

. E. S. Mahmoodian, Defining sets and uniqueness in graph colourings: a survey, submitted.

Problem 3 (BCC15.3). 1-track-less orientations. Posed by Jörg Zuther.

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A 1-*track* is a one-way infinite directed path (which may be directed either in or out).

Problem. Characterize those graphs which admit a 1-track-less orientation.

Note that every locally finite graph, and every m-partite graph (for finite m) has a 1-track-less orientation, but the countable complete graph does not.

Problem 4 (BCC15.4). Continuous maps between graphs. Posed by Anthony Hilton.

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A map is *k-to-1* if the inverse of every point in the codomain has cardinality k.

Problem. Determine the triples (k, m, n) for which there is a k-to-1 continuous map from $K_{m,m}$ to $K_{n,n}^{*}$, where these graphs are regarded as 1-dimensional simplicial complexes in the usual way (with edges homeomorphic to $[0, 1]$).

. A. J. W. Hilton, Initial and threshold values for exactly k-to-1 continuous maps between graphs, *Congressus Numerantium* 91 (1992), 254–270.

Problem 5 (BCC15.5). A generalization of Tarsi's problem. Posed by R. Klein and J. Schönheim.

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A graph is *m-degenerate* if every subgraph has a vertex of valency at most m.

Problem. Prove or disprove that a graph which is the edge-disjoint union of subgraphs $G_1, \ldots, G_s$, where G_i is m_i-degenerate, can be coloured with

$$\sum_{i=1}^{s} m_i + \left\lfloor \frac{1}{2}\left(1 + \sqrt{1 + 8 \sum_{1 \leq i < j \leq s} m_i m_j}\right)\right\rfloor$$

colours.

For $s = 2$, $m_1 = 1$, $m_2 = 2$, this is M. Tarsi's problem [1].

. R. Klein, On the colorability of m-composed graphs, *Discrete Math.* **133** (1994), 181–190.

Problem 6 (BCC15.6). Common vertices on longest paths. Posed by T. Gallai.

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Let G be a finite connected graph. Do any three longest paths in G have a common vertex? It is trivially true that every two longest paths have a common vertex; but there are graphs in which no vertex lies on all the longest paths.

. T. Gallai, Problem 4, in: P. Erdős, G. Katona, Eds., "Theory of graphs", Proc. Colloq. Tihany, Hungary, Sept. 1966, Academic Press, New York, 1968, p. 362.
. T. Zamfirescu, A two-connected planar graph without concurrent longest paths, *J. Combinatorial Theory* Ser. B **13** (1972), 116–121.
. S. Klavžar and M. Petkovšek, Graphs with nonempty intersection of longest paths, *Ars Combinatoria* 29 (1990), 43–52.

Problem 7 (BCC15.7). Edge-colourings of complete graphs. Posed by Peter Johnson.

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counter example

Suppose that the edges of the complete graph K_n ($n > 1$) are coloured with four colours R, G, B, Y such that each colour-class gives a connected subgraph on n vertices. It is easy to see from Satz 1.2(3) of Gallai [1] that at least three of the four triangles with edge colourings RGB, RGY, RBY, GBY occur.

Questions.

(a) Do all four occur?
(b) If not, how small can n be?
(c) What happens with more than four colours?

. T. Gallai, Transitiv Orientbare Graphen, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 25–66.

Problem 8 (BCC15.8). On the probability of connectedness. Posed by Peter Cameron.

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Which graphs G have the property that, in the class $\mathcal{X}(G)$ of graphs having no induced subgraph isomorphic to G, the limiting probability of connectedness is strictly between zero and one (in either the unlabelled or the labelled case)? (The smallest G with this property is the path of length 3; the probability of connectedness in $\mathcal{X}(G)$ is $\frac{1}{2}$ if the number of vertices is greater than one.)

Problem 9 (BCC15.9). Characteristic and chromatic polynomials. Posed by Roland Häggkvist.

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The *characteristic polynomial* of a graph G is the polynomial $\det(xI - A(G))$, where $A(G)$ is the adjacency matrix of G. Its roots are the *eigenvalues* of G.

Question. Are there more characteristic polynomials than chromatic polynomials of graphs on n vertices?

Problem 10 (BCC15.10). Graphs with three eigenvalues. Posed by Willem Haemers.

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Solved

Let G be a connected graph with just three distinct eigenvalues. Such a graph, if regular, must be strongly regular; and any strongly regular graph has this property. Non-regular examples include the complete bipartite graphs, and one further example on 36 vertices constructed by M. Muzychuk.

Questions.

(a) Is it true that G has at most two distinct valencies?
(b) Is G switching-equivalent to a null or strongly regular graph?
(c) Find more examples.

(The operation of *switching* a graph with respect to a set X of vertices replaces each edge from X to its complement by a non-edge and each such non-edge by an edge, leaving edges within or outside X unaltered.)

Editor's Note. M. Klin and M. Muzychuk [2] have pointed out that a family of examples were found in 1981 by Bridges and Mena [1], and have also constructed some 'sporadic' examples and re-formulated and analysed the question.

. W. G. Bridges and R. A. Mena, Multiplicative cones — a family of three eigenvalue graphs, *Aequat. Math.* 22 (1981), 208–214.
. M. Muzychuk and M. Klin, On graphs with three eigenvalues, in preparation.

Problem 11 (BCC15.11). Spectra of K_{s+1}-free graphs. Posed by Stephan Brandt.

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Let λ_1 and λ_n be the greatest and smallest eigenvalues of a graph on n vertices.

Conjecture. $(\lambda_1 + \lambda_n)/n \leq 4/25$ for any regular triangle-free graph on n vertices.

This conjecture would be true if any of the following two old Erdős conjectures holds (see e.g. [2]): Let G be a triangle-free graph on n vertices. Then (a) G contains a set of $\lfloor n/2 \rfloor$ vertices which span at most $n^2/50$ edges, and (b) G can be made bipartite by the omission of at most $n^2/25$ edges.

Problem. Let $\xi(s)$ be the supremum of $(\lambda_1 + \lambda_n)/n$ over the class of regular K_{s+1}-free graphs on n vertices. Determine or estimate $\xi(s)$.

The author [1] can show that

$$0.14 \leq \xi(2) \leq 3 - 2\sqrt{2} = 0.1715\ldots$$
$$(s-2)/s \leq \xi(s) \leq (s-2)/(s-1) \qquad \text{for } s \geq 3.$$

. S. Brandt, The local density of triangle-free graphs, Preprint 1995.
. P. Erdős, Problems and results in graph theory and combinatorial analysis, in: Proc. 5th British Combinatorial Conference (Aberdeen, 1975), Utilitas Math., Winnipeg, pp. 169–192.

Problem 12 (BCC15.12). Semiregular automorphism groups. Posed by Mikhail Klin.

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A permutation group is *semiregular* if no non-identity group element fixes a point. It is *regular* if it is transitive and semiregular. A graph is a Cayley graph if and only if its automorphism group contains a regular subgroup. It is known that there are vertex-transitive graphs which are not Cayley graphs (the smallest such being the Petersen graph.)

Question. Is there a vertex-transitive graph whose automorphism group contains no non-identity semiregular subgroup?

More generally, is there a 2-closed transitive permutation group containing no non-identity semiregular subgroup? (A permutation group is 2-*closed* if it is the automorphism group, preserving the colours, of some edge-coloured directed graph.)

Problem 13 (BCC15.13). A distance-regular graph. Posed by Leonard Soicher.

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Let C_{22} be the code obtained by puncturing the non-extended binary Golay code C_{23} in one coordinate. Then C_{22} is a [22, 12, 6] code with automorphism group $M_{22}:2$. Let M be the set of words of minimum non-zero weight in C_{22}, so that $|M| = 77$.

Let V be the set of pairs $\{v_1, v_2\}$ of words of C_{22} which satisfy $\mathrm{wt}(v_1) = \mathrm{wt}(v_2)$ and $v_1 + v_2 = \mathbf{1}$, where $\mathbf{1}$ is the all-1 word. Then $|V| = 672$. For $v = \{v_1, v_2\} \in V$, define

$$M(v) = \{m \in M | \mathrm{wt}(v_1 + m) = \mathrm{wt}(v_2 + m)\}.$$

Then $|M(v)| = 55$ for all $v \in V$.

Define a graph Γ with vertex set V, in which $v \sim w$ if and only if $|M(v) \cap M(w)| = 43$. Then Γ is a distance-regular, but not distance-transitive graph. Moreover, the distance function in Γ is given by

$$d_\Gamma(v, w) = \frac{1}{4}(47 - |M(v) \cap M(w)|)$$

for $v, w \in V$, $v \neq w$.

These facts have been proved using the package **GRAPE** (see [1]).

Problem. (a) Prove this by hand, to help understand Γ.

(b) Can a similar construction be applied to other codes with even length and minimum weight, to construct other distance-regular graphs?

. L. H. Soicher, Yet another distance-regular graph related to a Golay code, *Electronic J. Combinatorics* 2 (1995), #N1.

Problem 14 (BCC15.14). Pasch configurations in 3-hypergraphs. Posed by G. B. Khosrovshahi.

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Let $X = \{1, 2, \ldots, v\}$. Denote the set of all 3-subsets of X by $P_3(X)$. Show that for $v \geq 6$, any $(\binom{v}{2}+1)$-subset of $P_3(X)$ must contain a *Pasch configuration*, that is, $\{abc, axy, bxz, cyz\}$ for some $a, b, c, x, y, z \in X$.

Problem 15 (BCC15.15). Critical sets in Latin squares. Posed by Ebad Mahmoodian.

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A *critical set* in an $n \times n$ array with entries from the set $\{1, \ldots, n\}$ is a set S of the positions of the array with the property that the entries in the positions of S have a unique extension to a Latin square of order n.

Problem. Show that any critical set in a Latin square of order n has cardinality at least $\lfloor n^2/4 \rfloor$.

Problem 16 (BCC15.16). Loops with conditions on area. Posed by Alain Valette.

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A *loop* γ is a closed trajectory in the square lattice. Its *algebraic area* is $A(\gamma) = \oint_\gamma x dy$. Let $N(2k;\mathcal{P})$ be the number of loops based at 0, of length $2k$, which satisfy property $\mathcal{P}$. So, for example, $N(2k;\emptyset) = \binom{2k}{k}^2$.

Problem. Find either formulae or asymptotics for $N(2k; A = l)$ and $N(2k; A \equiv l \pmod q)$, for given l, q.

For example, $\lim_{k\to\infty} N(2k; A = 0)^{1/2k} = 4$. Also, if $n(2k, l, q)$ denotes $N(2k; A \equiv l \pmod q)$, then it is known that

$\lim_{k\to\infty}(n(2k;0,2) - n(2k;1,2))^{1/2k} = 2\sqrt{2}$,
$\lim_{k\to\infty}(n(2k;0,3) - n(2k;1,3))^{1/2k} = 1 + \sqrt{3}$,
$\lim_{k\to\infty}(n(2k;0,4) - n(2k;2,4))^{1/2k} = 2\sqrt{2}$.

This problem (secretly) deals with the walk generating function of the discrete Heisenberg group in its 2-generator presentation.

Problem 17 (BCC15.17). Proof of an identity. Posed by Richard Lewis.

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For complex numbers $z \neq 0$, $|w| < 1$, set $[z;w] = \prod_{n=1}^{\infty}(1 - zw^{n-1})(1 - z^{-1}w^n)$. It can be shown, using Cauchy's theorem, that for any non-zero complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $a_1 \ldots a_n = b_1 \ldots b_n$, and any q with $|q| < 1$,

$$\sum_{r=1}^{n} \frac{[a_1 b_r^{-1}; q][a_2 b_r^{-1}; q] \ldots [a_n b_r^{-1}; q]}{[b_1 b_r^{-1}; q][b_2 b_r^{-1}; q] \ldots \hat{} \ldots [b_n b_r^{-1}; q]} = 0,$$

where the $\hat{}$ means to omit the term $[b_r b_r^{-1}; q]$.

Problem. Find a combinatorial (bijective) proof of this inequality.

Problem 18 (BCC15.18). Counting classes of graphs. Posed by Peter Cameron.

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Find good asymptotic estimates for the numbers of

(a) line graphs,
(b) line graphs of bipartite graphs,
(c) comparability graphs of 2-dimensional posets

on n vertices? (The last class of graphs are defined as follows: Take a permutation π of $\{1, \ldots, n\}$, and join i to j whenever $(i - j)(i\pi - j\pi) > 0$.)

Problem 19 (BCC15.19). Antichains in products of chains. Posed by Jonathan D. Farley.

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Let $\theta(P)$ be the set of antichains of the poset P, and let $\underline{n}$ be the n-element chain. Dedekind's problem [2] asks for the value of $|\theta(\underline{2}^n)|$. It is easy to show that $|\theta(\underline{n})| = n+1$ and $|\theta(\underline{m} \times \underline{n})| = \binom{m+n}{m}$. MacMahon, Stanley [3], [4], Berman and Köhler [1] showed that

$$|\theta(\underline{k} \times \underline{m} \times \underline{n})| = \prod_{j=0}^{k-1} \binom{m+n+j}{m} \Big/ \binom{m+j}{m}.$$

(Despite appearances, this function is symmetric!)

Problem. What is $|\theta(\underline{j} \times \underline{k} \times \underline{m} \times \underline{n})|$?

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Problem 20 (BCC15.20). Cycles of a permutation. Posed by Peter Cameron.

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As an example of a "typical" automorphism of the space of periodic integrable functions (acting on Fourier coefficients), W. Rudin [1] considered the permutation of the integers defined by

$$3n \mapsto 2n, \qquad 3n+1 \mapsto 4n+1, \qquad 3n-1 \mapsto 4n-1.$$

Problem. Describe the cycles of this permutation. In particular, does it have only finitely many finite cycles?

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Problem 21 (BCC15.21). Combinatorics and control theory. Posed by Holger Schellwat.

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In place of the Laplace transform, which is used to model continuous time control systems, in discrete control the $\mathcal{Z}$-transform is the basic tool. For a function $f : \mathbb{Z} \to \mathbb{C}$, its $\mathcal{Z}$-transform is defined by $\mathcal{Z}(f)(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$. One problem, for example, is the stability of an inert linear controller and is determined by the loci of poles of a fraction of polynomials in $\mathbb{R}[x]$, constituting the $\mathcal{Z}$-transform of the transfer function [1]. On the other hand, the method of generating functions [2] is used widely in combinatorics to solve enumeration problems. If $(a_i : i \in \mathbb{N})$ is a sequence of numbers, for instance counting the number of distinct combinatorial objects of a certain kind, its associated ordinary generating function is the formal power series $\sum_{n=0}^{\infty} a_n z^n$. But up to the sign of the exponent, this is the defining sum for the $\mathcal{Z}$-transform of the sequence, viewed as a function. Thus it seems natural to explore the implications of this correspondence. Is it even possible to use it to translate problems in control theory into problems in combinatorics and/or vice versa? Could representation theory help to establish such a correspondence?

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