The spectral radius of connected graphs with the independence number

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This work is joint with my advisor Xiao-Dong Zhang.

1. Basic definition and background
2. The related theorem
3. Open problems

A simple graph $G = (V(G), E(G))$, vertex set $V = \{1, 2, ..., n\}$.
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Adjacency matrix: $A(G) = (a_{ij})$, where $a_{ij} = 1$ if and only if $i \sim j$.

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signless Laplacian matrix: $Q(G) = D(G) + A(G)$.

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Turán Graph $T_{n,t}$: A complete t-partite graph, the cardinality of each part is $\lfloor \frac{n}{t} \rfloor$ or $\lceil \frac{n}{t} \rceil$.

Turán Theorem (1941, P. Turán)

Let G be a graph of order n not containing K_{t+1}. If $n = kt + r, 0 \leq r < t$, then

$$e(G) \leq e(T_{n,t}) = \frac{t^2 - t}{2} k^2 + (t-1)rk + \frac{r(r-1)}{2}.$$

Moreover, equality holds if and only if $G = T_{n,t}$.

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$i(G)$: max$\{s$: G has s-independent set$\}$.
$w(G)$: max$\{s$: G has s-clique$\}$.

Expanded Turán Theorem (1978, B. Bollobás)

Let G be a graph of order n with clique number w. Then

$$k_s(G) \leq k_s(T_{n,t}),\ for\ 2 \leq s \leq w.$$

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Problem 1: How about the relation between $e(G)$ and the spectral radius of the adjacency matrix $A(G)$?

For the adjacency matrix $A(G)$ of G and the largest eigenvalue $\mu(G)$, by Rayleigh quotient,

$$\frac{2e(G)}{n} \leq \mu(G) \Longrightarrow e(G) \leq \frac{n\mu(G)}{2}.$$

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Spectral Turán Theorem (1998, B. Guiduli, 2007, V. Nikiforov)

Let G be a simple graph of order n not containing K_{t+1} as a subgraph. Then

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Spectral Turán Theorem (2013, B. He, X.-D. Zhang, Y.-L. Jin)

Let G be a simple graph of order n not containing $K_{t+1},\ t > 2$ as a subgraph. Then

$$q(G) \leq q(T_{n,t})$$

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For a simple graph G,

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Turán Theorem (1941, P. Turán)

Let G be a graph of order n not containing $(t+1)$-independent set. Then

$$e(G) \geq e(T_{n,t}^c).$$

Moreover, equality holds if and only if $G = T_{n,t}^c$.

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Theorem (2011 V. Nikiforov)

Let G be a simple graph and $i(G) = \alpha$, then $\mu(G) \geq \lceil \frac{n}{\alpha} \rceil - 1$.

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Problem 4: Consider any connected graph G of order n and $i(G) = \alpha$, whether there is a graph H such that

$$\mu(G) \geq \mu(H)$$

the equality holds if and only if $G = H$?

Theorem (2009 Xu, Hong, Shu and Zhai)

Let G be a connected graph of order n with the independence number α. If $\alpha \in \{1, 2, \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, n-3, n-2, n-1\}$, then the extremal graphs with smallest spectral radius have been characterized.

Theorem (2012 Du and Shi)

Let G be a connected graph of order $n = k\alpha \geq 108$ with the independence number α. If $\alpha = 3$ or 4, then

$$\mu(G) \geq \mu(P_{n,\alpha}),$$

where graph $P_{n,\alpha}$ is obtained from a path of order α by replacing each vertex to a clique of order k and has exactly $2\alpha - 2$ cut vertices.

Let

$$\mathcal{T}_{n,\alpha} = \{ \text{ all clique trees of order } n \ \},$$

$\mathcal{G}_{n,\alpha} = \{$ all connected graphs of order n, the independence number $\alpha\}$.

where the clique trees are obtained from a tree of order α by replacing each vertex to a clique of order $\lfloor \frac{n}{\alpha} \rfloor$ or $\lceil \frac{n}{\alpha} \rceil$.

Theorem (2014 Jin and Zhang)

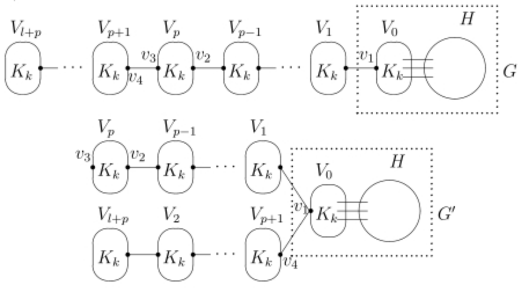
If $n = k\alpha$ and $k > \frac{17\alpha+15}{8}$, then $P_{n,\alpha}$ is the only graph having the minimum spectral radius in $\mathcal{G}_{n,\alpha}$. In other words, for any $G \in \mathcal{G}_{n,\alpha}$, $\mu(G) \geq \mu(P_{n,\alpha})$ with equality if and only if $G = P_{n,\alpha}$.

Lemma (2014 Jin and Zhang)

Let $n = k\alpha$ and $k > \frac{17\alpha+15}{8}$. If a connected graph G has the minimum spectra radius among all graphs in $\mathcal{G}_{n,\alpha}$, then $G \in \mathcal{T}_{n,\alpha}$.

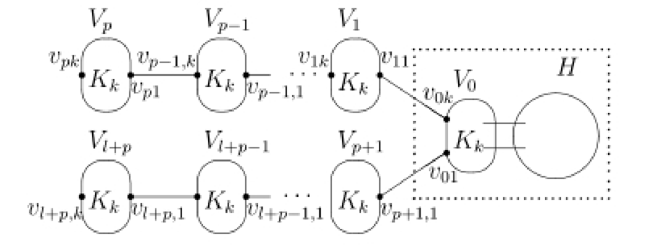
Lemma (2014 Jin and Zhang)

Let $n = k\alpha > 2\alpha$ and $G \in \mathcal{T}_{n,\alpha}$ be a graph obtained by joining an edge from a non-cut vertex of a graph $H \in \mathcal{T}_{n-k(l+p),\alpha-(l+p)}$ and a non-cut vertex of $P_{k(l+p),l+p}$. Let G' be the graph obtained from G by deleting the edge v_3v_4 and adding edge v_1v_4. If G has an induced subgraph $P_{k(2l+p),(2l+p)}$ containing v_1, then $\mu(G') > \mu(G)$.



Lemma (2014 Jin and Zhang)

Let $n = k\alpha > 2\alpha$, $H_{p,l} \in \mathcal{T}_{n,\alpha}$ be a graph obtained by joining two edges from two non-cut vertices of a graph $H \in \mathcal{T}_{n-k(l+p),\alpha-(l+p)}$ with a non-cut vertex of $P_{kp,p}$ and a non-cut vertex of $P_{kl,l}$, where $H \neq K_k$ and $p \geq l \geq 1$. Then $\psi(H_{p,l}, x) < \psi(H_{p+1,l-1}, x)$ for $x \geq \mu(H_{p+1,l-1})$. Furhter $\mu(H_{p,l}) > \mu(H_{p+1,l-1})$.



Conjecture(2012 Du and Shi)

Conjecture. *For* $k, n, \alpha \in \mathbb{N}$, *let* $G \in \mathcal{G}_{n,\alpha}$ *have the minimum spectral radius in* $\mathcal{G}_{n,\alpha}$. *Then for sufficiently large* n,

(1) $G \cong F(k, k, k+1)$ *for* $\alpha = 3$, $n = 3k+1$,
(2) $G \cong F(k+1, k, k+1)$ *for* $\alpha = 3$ *and* $n = 3k+2$,
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(5) $G \cong F(k+1, k, k+1, k+1)$ *for* $\alpha = 4$ *and* $n = 4k+3$.

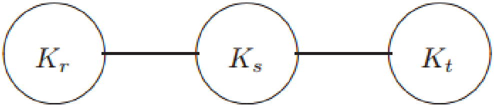

Fig. 1. The graph $F(r, s, t)$.

THANKS FOR YOUR ATTENTIONS!