Diffy 六邊形之探討
A Study about Diffy Hexagons

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Outline

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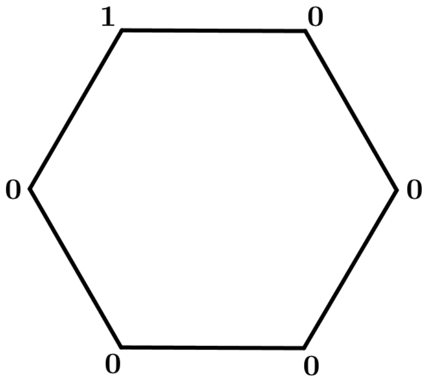
Introduction

Diffy Hexagons

The Diffy Hexagons which are generalized Diffy Boxes are games with the following procedures:

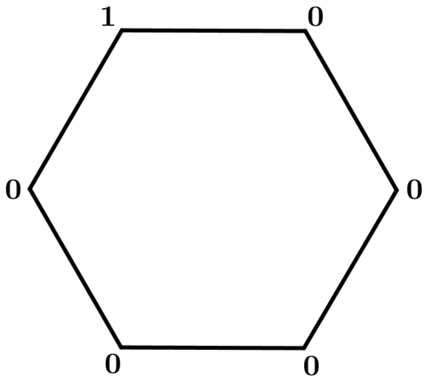
Step 1

Arrange six nonnegative integers around a regular hexagon.



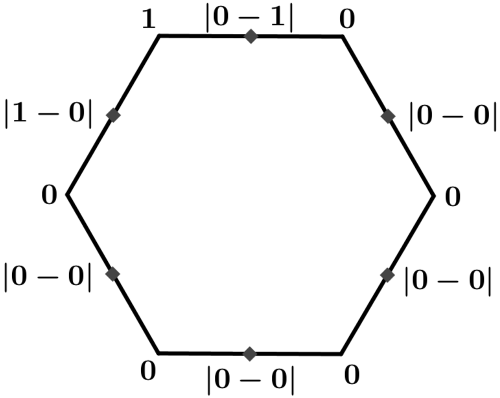
Step 2

Produce another regular hexagon of six nonnegative integers from the one obtained in Step 1:

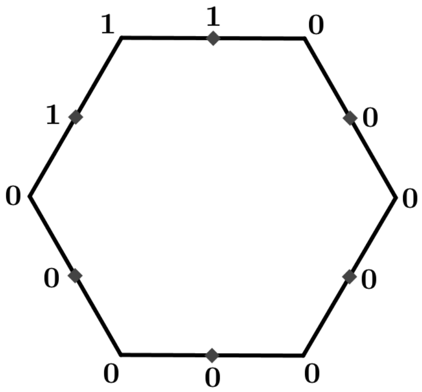


Step 2-1

For each adjacent pair of numbers, compute the absolute value of their difference and place it between them.

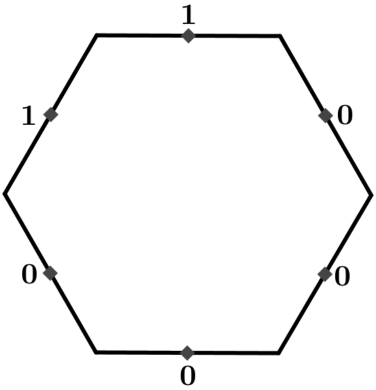


Step 2-1



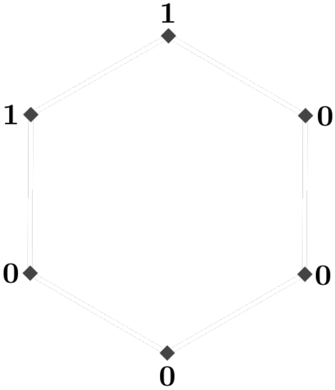
Step 2-2

Remove the original numbers.



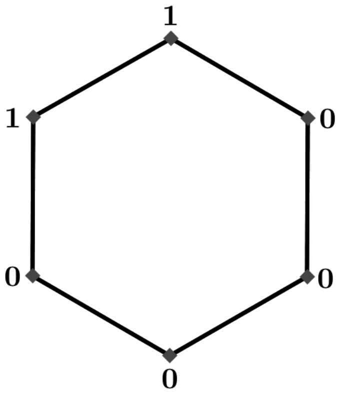
Step 2-3

Remove the original regular hexagon.



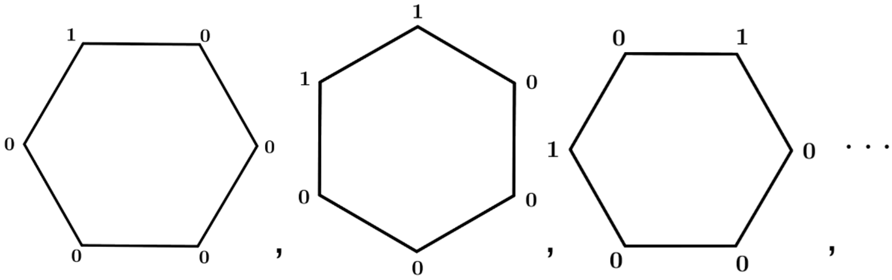
Step 2-4

Form the new regular hexagon with the remaining numbers.



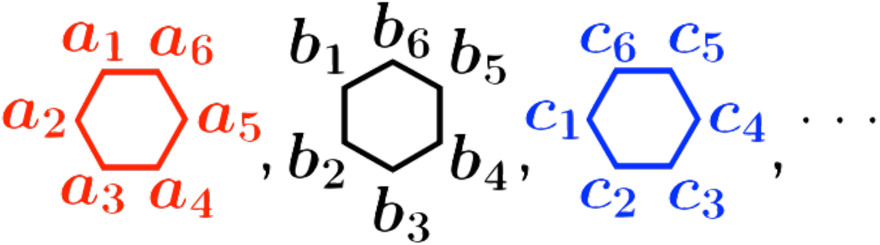
Step 3

To obtain a sequence of regular hexagons of six nonnegative integers by performing Step 2 over and over.



Notations

Without loss of generality, we denote a regular hexagon of six nonnegative integers as follows:



where $b_i = |a_i - a_{i+1}|$, $c_i = |b_i - b_{i+1}|$, $b_6 = |a_6 - a_1|$, and $c_6 = |b_6 - b_1|$, $i = 1, 2, \cdots, 5$.

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Define $D : A_N \to A_N$ by

$$D(a_1, a_2, \cdots, a_N) = (\,|a_1 - a_2|, \cdots, |a_{N-1} - a_N|, |a_N - a_1|\,)$$

for all $(a_1, a_2, \cdots, a_N) \in A_N$.

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for all $(a_1, a_2, \cdots, a_N) \in A_N$.
Then, D is a well-defined function.

Ducci processes

Definition (1.1)

The function $D : A_N \to A_N$ defined by

$$D(a_1, a_2, \cdots, a_N) = (\,|a_1 - a_2|, \cdots, |a_{N-1} - a_N|, |a_N - a_1|\,)$$

for all $(a_1, a_2, \cdots, a_N) \in A_N$ is called a *Ducci process*.

Ducci sequences of N-tuples in A_N

Definition (1.2)

Let $\vec{a} = (a_1, a_2, \cdots, a_N) \in A_N$. A sequence of the form that $\vec{a}, D(\vec{a}), D^2(\vec{a}), \cdots$ is called the *Ducci sequence of* $\vec{a}$.

Ducci sequences of N-tuples in A_N

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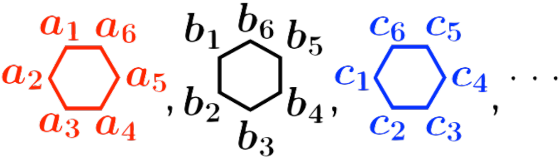
Diffy Hexagon games

Remark (1.3)

Note that a 6-tuple of nonnegative integers is regarded as written in a regular hexagon, and hence a Ducci sequence of 6-tuples in A_6 is regarded as a sequence of regular hexagons, that is, a Diffy Hexagon game.



$$D(a_1, a_2, \cdots, a_6) = (b_1, b_2, \cdots, b_6)$$
$$D(b_1, b_2, \cdots, b_6) = (c_1, c_2, \cdots, c_6)$$

Ducci Sequences

Existence of the period of Ducci sequences

Lemma (2.1)

Let $\vec{a} \in A_N$. Then, there are nonnegative integers n, k with $n > k$ such that $D^n(\vec{a}) = D^k(\vec{a})$.

Proof

Existence of the period of Ducci sequences

Lemma (2.1)

Let $\vec{a} \in A_N$. Then, there are nonnegative integers n, k with $n > k$ such that $D^n(\vec{a}) = D^k(\vec{a})$.

Proof

Example

$$\begin{aligned}
\vec{a} &= (1,0,0,2,1,0) \\
D(\vec{a}) &= (1,0,2,1,1,1) \\
D^2(\vec{a}) &= (1,2,1,0,0,0) \\
D^3(\vec{a}) &= (1,1,1,0,0,1) \\
D^4(\vec{a}) &= (0,0,1,0,1,0) \\
D^5(\vec{a}) &= (0,1,1,1,1,0)
\end{aligned}$$

$$\begin{aligned}
D^6(\vec{a}) &= (1,0,0,0,1,0) \\
D^7(\vec{a}) &= (1,0,0,1,1,1) \\
D^8(\vec{a}) &= (1,0,1,0,0,0) \\
D^9(\vec{a}) &= (1,1,1,0,0,1) \\
&= D^3(\vec{a})
\end{aligned}$$

The period and cycle of Ducci sequences

Definition (2.2)

Let $\vec{a} \in A_N$. Suppose that n is the positive integer such that $\vec{a}, D(\vec{a}), D^2(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct and $D^n(\vec{a}) = D^k(\vec{a})$, where $0 \leq k \leq n-1$.

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We define the *period of* $\vec{a}$ to be $n-k$ and the $(n-k)$*-cycle of* $\vec{a}$ (or simply the *cycle of* $\vec{a}$) to be $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$.

The largest component of N-tuples in A_N

Definition (2.3)

Let $\vec{a} \in A_N$. The the largest component of $\vec{a}$ is denoted by $\max \vec{a}$.

A property about the largest component of N-tuples in A_N

Lemma (2.5)

Let $\vec{a} \in A_N$. For all nonnegative integers r, s with $r \geq s$, then we have $\max D^r(\vec{a}) \leq \max D^s(\vec{a})$.

Proof

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Proof

Example

$$\begin{aligned}
\vec{a} &= (1,0,0,2,1,0) & \max \vec{a} &= 2 \\
D(\vec{a}) &= (1,0,2,1,1,1) & \max D(\vec{a}) &= 2 \\
D^2(\vec{a}) &= (1,2,1,0,0,0) & \max D^2(\vec{a}) &= 2 \\
D^3(\vec{a}) &= (1,1,1,0,0,1) & \max D^3(\vec{a}) &= 1 \\
D^4(\vec{a}) &= (0,0,1,0,1,0) & \max D^4(\vec{a}) &= 1 \\
&\vdots & &\vdots
\end{aligned}$$

The largest component of N-tuples in the cycle

Lemma (2.6)

Let $\vec{a} \in A_N$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$. Then,

$$\max D^r(\vec{a}) = \max D^s(\vec{a}), \forall\, k \leq r, s \leq n-1.$$

Proof

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Proof

Example

$$\vec{a} = (0,1,2,2,1,0)$$
$$D(\vec{a}) = (1,1,0,1,1,0)$$
$$D^2(\vec{a}) = (0,1,1,0,1,1)$$
$$D^3(\vec{a}) = (1,0,1,1,0,1)$$
$$D^4(\vec{a}) = (1,1,0,1,1,0) = D(\vec{a})$$

Components of N-tuples in the cycle

Theorem (2.12)

Let $\vec{a} \in A_N$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$.

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Let $\vec{a} \in A_N$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$. Then, the components of $D^i(\vec{a})$ are all equal to either 0 or M for each $i = k, k+1, \cdots, n-1$, where $M = \max D^k(\vec{a})$.

Proof

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Proof

Example

$$\vec{a} = (0,2,4,4,2,0) \qquad D^3(\vec{a}) = (2,0,2,2,0,2)$$

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The converse of Theorem 2.12 fails in general

Remark (2.13)

If $N \neq 2$, then there are $\vec{a}, \vec{b} \in A_N$ with $D(\vec{a}) = \vec{b}$ such that

$$\max \vec{a} = \max \vec{b} = M$$

The converse of Theorem 2.12 fails in general

Remark (2.13)

If $N \neq 2$, then there are $\vec{a}, \vec{b} \in A_N$ with $D(\vec{a}) = \vec{b}$ such that

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and the components of $\vec{a}$, $\vec{b}$ aren't all equal to either 0 or M.

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Example

Let $N = 6$, $\vec{a} = (2014, 0, 1, 1, 1, 1)$ and $D(\vec{a}) = \vec{b}$

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and the components of $\vec{a}$, $\vec{b}$ aren't all equal to either 0 or M.

Proof

Example

Let $N = 6$, $\vec{a} = (2014, 0, 1, 1, 1, 1)$ and $D(\vec{a}) = \vec{b}$
Then, $\vec{b} = (2014, 1, 0, 0, 0, 2013)$

The greatest common divisor of components of N-tuples in A_N

Definition (2.14)

Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$.

The greatest common divisor of components of N-tuples in A_N

Definition (2.14)

Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$. If $\vec{a} = (a_1, a_2, \cdots, a_N)$, then $\gcd \vec{a}$ is the number $\gcd(a_1, a_2, \cdots, a_N)$.

The relation between $\max \vec{a}$ and $\gcd \vec{a}$

Lemma (2.15)

Let $\vec{a} \in A_N$ *with* $\vec{a} \neq \vec{\mathbf{0}}$ *and* n *be a nonnegative integer. Then, we obtain that* $\gcd \vec{a} \mid \max D^n(\vec{a})$.

Proof

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Proof

Example

Let $\vec{a} = (0, 2, 4, 4, 2, 0)$

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Proof

Example

Let $\vec{a} = (0, 2, 4, 4, 2, 0)$

$\implies \gcd(\vec{a}) = 2$

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Proof

Example

Let $\vec{a} = (0, 2, 4, 4, 2, 0)$

$\implies \gcd(\vec{a}) = 2$

$$D(\vec{a}) = (2, 2, 0, 2, 2, 0)$$

$$D^2(\vec{a}) = (0, 2, 2, 0, 2, 2)$$

$$D^3(\vec{a}) = (2, 0, 2, 2, 0, 2)$$

$$D^4(\vec{a}) = (2, 2, 0, 2, 2, 0) = D(\vec{a})$$

Components of N-tuples in the cycle

Corollary (2.16)

Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$.

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Proof

In Corollary 2.16, M may be any nonnegative integer

Example (2.17)

Let $d = \gcd \vec{a}, K \geq 1$ be integers and $\vec{a} = (d, d, d, d, d, Kd) \in A_6$.

In Corollary 2.16, M may be any nonnegative integer

Example (2.17)

Let $d = \gcd \vec{a}$, $K \geq 1$ be integers and $\vec{a} = (d, d, d, d, d, Kd) \in A_6$.

$$\begin{aligned}
D(\vec{a}) &= (0, 0, 0, 0, (K-1)d, (K-1)d) \\
D^2(\vec{a}) &= (0, 0, 0, (K-1)d, 0, (K-1)d) \\
D^3(\vec{a}) &= (0, 0, (K-1)d, (K-1)d, (K-1)d, (K-1)d) \\
D^4(\vec{a}) &= (0, (K-1)d, 0, 0, 0, (K-1)d) \\
D^5(\vec{a}) &= ((K-1)d, (K-1)d, 0, 0, (K-1)d, (K-1)d) \\
D^6(\vec{a}) &= (0, (K-1)d, 0, (K-1)d, 0, 0) \\
D^7(\vec{a}) &= ((K-1)d, (K-1)d, (K-1)d, (K-1)d, 0, 0) \\
D^8(\vec{a}) &= (0, 0, 0, (K-1)d, 0, (K-1)d) = D^2(\vec{a}) \\
M &= (K-1)d
\end{aligned}$$

Similar Cycles

Similar cycles

Definition (3.1)

Let $\vec{a} \in A_N$ and $\vec{b} \in (\mathbb{Z}_2)^N$. Suppose that $D^k(\vec{b}), D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ is the $(n-k)$-cycle of $\vec{b}$.

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The period of similar cycles

Theorem (3.2)

Let $\vec{a} \in A_N$. Then, the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$, where $\vec{b} \in (\mathbb{Z}_2)^N$ and the period of $\vec{b}$ is equal to the period of $\vec{a}$.

Proof

The period of similar cycles

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Let $\vec{a} \in A_N$. Then, the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$, where $\vec{b} \in (\mathbb{Z}_2)^N$ and the period of $\vec{b}$ is equal to the period of $\vec{a}$.

Proof

Example

$$\begin{aligned} \vec{a} &= (0,2,4,4,2,0) \\ D(\vec{a}) &= (2,2,0,2,2,0) \\ D^2(\vec{a}) &= (0,2,2,0,2,2) \\ D^3(\vec{a}) &= (2,0,2,2,0,2) \\ D^4(\vec{a}) &= (2,2,0,2,2,0) \\ &= D(\vec{a}) \end{aligned}$$

$$\begin{aligned} \vec{b} &= (0,1,0,0,1,0) \\ D(\vec{b}) &= (1,1,0,1,1,0) \\ D^2(\vec{b}) &= (0,1,1,0,1,1) \\ D^3(\vec{b}) &= (1,0,1,1,0,1) \\ D^4(\vec{b}) &= (1,1,0,1,1,0) \\ &= D(\vec{b}) \end{aligned}$$

Cycles of N-tuples in A_N

Remark (3.3)

When we discuss cycles of N-tuples in A_N, it is enough to cope with N-tuples in $(\mathbb{Z}_2)^N$ according to Theorem 3.2.

Periods of N-tuples in A_N

Theorem (3.13)

Let $\vec{e}_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{iN}) \in A_N$*, where* δ_{ij} *is the Kronecker delta for all* $i, j \in \{1, 2, \cdots, N\}$.

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(a) *If* $D^r(\vec{e}_1) = D^s(\vec{e}_1)$ *for some nonnegative integers* r, s

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(a) *If* $D^r(\vec{e}_1) = D^s(\vec{e}_1)$ *for some nonnegative integers* r, s, *then we have:* $D^r(\vec{b}) = D^s(\vec{b}), \forall \vec{b} \in (\mathbb{Z}_2)^N$.

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(a) *If* $D^r(\vec{e}_1) = D^s(\vec{e}_1)$ *for some nonnegative integers* r, s*, then we have:* $D^r(\vec{b}) = D^s(\vec{b}), \forall\, \vec{b} \in (\mathbb{Z}_2)^N$.

(b) *The period of* $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N$ *are all identical.*

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(a) *If $D^r(\vec{e}_1) = D^s(\vec{e}_1)$ for some nonnegative integers r, s, then we have: $D^r(\vec{b}) = D^s(\vec{b}), \forall\, \vec{b} \in (\mathbb{Z}_2)^N$.*

(b) *The period of $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N$ are all identical.*

(c) *If $\vec{a} \in A_N$, then the period of $\vec{a}$ divides the period of $\vec{e}_1$.*

Periods of N-tuples in A_N

Theorem (3.13)

Let $\vec{e}_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{iN}) \in A_N$, where δ_{ij} is the Kronecker delta for all $i, j \in \{1, 2, \cdots, N\}$. Then, we have:

(a) *If $D^r(\vec{e}_1) = D^s(\vec{e}_1)$ for some nonnegative integers r, s, then we have: $D^r(\vec{b}) = D^s(\vec{b}), \forall\, \vec{b} \in (\mathbb{Z}_2)^N$.*

(b) *The period of $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N$ are all identical.*

(c) *If $\vec{a} \in A_N$, then the period of $\vec{a}$ divides the period of $\vec{e}_1$. In particular, the maximal period of N-tuples in A_N is equal to the period of $\vec{e}_1$.*

Proof

Cycles of 2^r-tuples in A_{2^r}

Theorem (3.15)

Let r be a positive integer. Suppose that $N = 2^r$.

Cycles of 2^r-tuples in A_{2^r}

Theorem (3.15)

Let r be a positive integer. Suppose that $N = 2^r$. If $\vec{a} \in A_N$, then the cycle of $\vec{a}$ is similar to the 1-cycle of $\vec{\mathbf{0}}$.

Proof

Cycles of 2^r-tuples in A_{2^r}

Theorem (3.15)

Let r be a positive integer. Suppose that $N = 2^r$. If $\vec{a} \in A_N$, then the cycle of $\vec{a}$ is similar to the 1-cycle of $\vec{\mathbf{0}}$.

Proof

Example

$$\begin{aligned} \vec{a} &= (0,1,2,0) \\ D(\vec{a}) &= (1,1,2,0) \\ D^2(\vec{a}) &= (0,1,2,1) \\ D^3(\vec{a}) &= (1,1,1,1) \end{aligned}$$

$$\begin{aligned} D^4(\vec{a}) &= (0,0,0,0) \\ D^5(\vec{a}) &= (0,0,0,0) \\ &= D^4(\vec{a}) \end{aligned}$$

Diffy Hexagons

Introduction

According to Remark 1.3, we shall concentrate on the cycles of 6-tuples in A_6 in this chapter.

The period of 6-tuples in A_6

Theorem (4.1)

The period of 6-tuples in A_6 divides 6.

The period of 6-tuples in A_6

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Proof

The period of 6-tuples in A_6

Theorem (4.1)

The period of 6-tuples in A_6 divides 6. In particular, the maximal period of 6-tuples in A_6 is equal to 6.

Proof

Example

$$\vec{e}_1 = (1,0,0,0,0,0)$$
$$D(\vec{e}_1) = (1,0,0,0,0,1)$$
$$D^2(\vec{e}_1) = (1,0,0,0,1,0)$$
$$D^3(\vec{e}_1) = (1,0,0,1,1,1)$$
$$D^4(\vec{e}_1) = (1,0,1,0,0,0)$$
$$D^5(\vec{e}_1) = (1,1,1,0,0,1)$$
$$D^6(\vec{e}_1) = (0,0,1,0,1,0)$$
$$D^7(\vec{e}_1) = (0,1,1,1,1,0)$$
$$D^8(\vec{e}_1) = (1,0,0,0,1,0) = D^2(\vec{e}_1)$$

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

If $\vec{b} \in (\mathbb{Z}_2)^6$, then the cycle of $\vec{b}$ is one of the followings:

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

If $\vec{b} \in (\mathbb{Z}_2)^6$, then the cycle of $\vec{b}$ is one of the followings:

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$

$$\mathbf{(0, 0, 0, 0, 0, 0)} \circlearrowleft$$

$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

If $\vec{b} \in (\mathbb{Z}_2)^6$, then the cycle of $\vec{b}$ is one of the followings:

(i) (1-cycle) $(0,0,0,0,0,0)$

$$\begin{tikzcd}[column sep=small]
(0,1,0,1,0,1) \arrow[dr] & & (1,0,1,0,1,0) \arrow[dl] \\
& (1,1,1,1,1,1) \arrow[d] & \\
& \mathbf{(0,0,0,0,0,0)} \arrow[loop below] &
\end{tikzcd}$$

$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(ii) (3-cycle) $(0,1,1,0,1,1),(1,0,1,1,0,1),(1,1,0,1,1,0)$

$$\begin{tikzcd}
& \mathbf{(0,1,1,0,1,1)} \arrow[dr, bend left] & \\
\mathbf{(1,1,0,1,1,0)} \arrow[ur, bend left] & & \mathbf{(1,0,1,1,0,1)} \arrow[ll, bend left]
\end{tikzcd}$$

$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(ii) (3-cycle) $(0,1,1,0,1,1), (1,0,1,1,0,1), (1,1,0,1,1,0)$

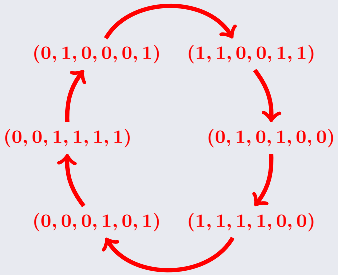
$$
\begin{tikzcd}[column sep=tiny]
 & & (0,0,0,1,1,1) \arrow[dr] & & (1,1,1,0,0,0) \arrow[dl] & & \\
 & & & (0,0,1,0,0,1) \arrow[d] & & & \\
 & & & \mathbf{(0,1,1,0,1,1)} \arrow[dr, bend left] & & & \\
 & & \mathbf{(1,1,0,1,1,0)} \arrow[ur, bend left] & & \mathbf{(1,0,1,1,0,1)} \arrow[ll, bend left] & & \\
 & (0,1,0,0,1,0) \arrow[ur] & & & & (1,0,1,1,0,0) \arrow[ul] & \\
(1,1,0,0,0,1) \arrow[ur] & & (0,0,1,1,1,0) \arrow[ul] & & (0,1,1,1,0,0) \arrow[ur] & & (1,0,0,0,1,1) \arrow[ul]
\end{tikzcd}
$$

$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(iii) (6-cycle) $(0,1,0,0,0,1), (1,1,0,0,1,1), (0,1,0,1,0,0),$
$(1,1,1,1,0,0), (0,0,0,1,0,1), (0,0,1,1,1,1)$

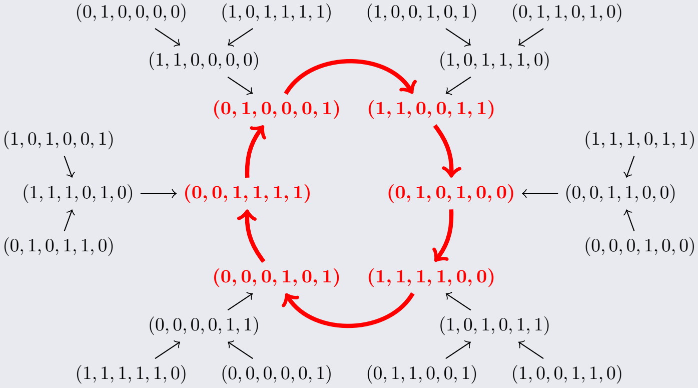


$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(iii) (6-cycle) $(0,1,0,0,0,1), (1,1,0,0,1,1), (0,1,0,1,0,0), (1,1,1,1,0,0), (0,0,0,1,0,1), (0,0,1,1,1,1)$

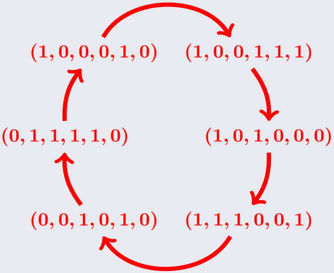


$\longrightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(iv) (6-cycle) $(1,0,0,0,1,0), (1,0,0,1,1,1), (1,0,1,0,0,0),$
$(1,1,1,0,0,1), (0,0,1,0,1,0), (0,1,1,1,1,0)$

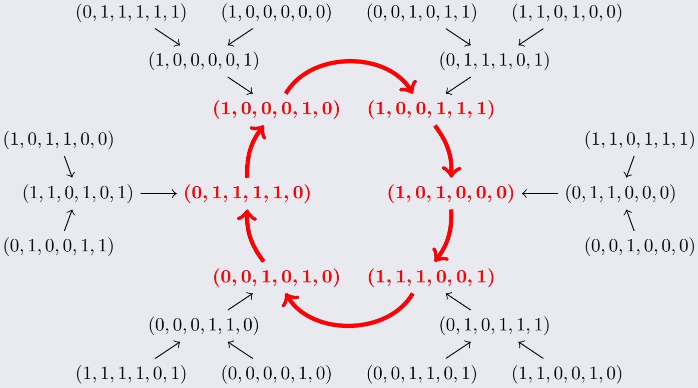


$\rightarrow$: *a Ducci process*

Cycles of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.2)

(iv) (6-cycle) $(1,0,0,0,1,0), (1,0,0,1,1,1), (1,0,1,0,0,0), (1,1,1,0,0,1), (0,0,1,0,1,0), (0,1,1,1,1,0)$



$\longrightarrow$: *a Ducci process*

Ducci sequences and Diffy Hexagons

As in Remark 1.3, a 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ in A_6 is regarded as written in a regular hexagon.

Ducci sequences and Diffy Hexagons

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However, regular hexagons have symmetries under rotations and reflections,

Ducci sequences and Diffy Hexagons

As in Remark 1.3, a 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ in A_6 is regarded as written in a regular hexagon.
However, regular hexagons have symmetries under rotations and reflections, but $(a_1, a_2, a_3, a_4, a_5, a_6)$ does not.

Prepare for an identification on Ducci sequences

Write $\mathcal{D}_6 = \{(1)(2)(3)(4)(5)(6), (123456), (135)(246), (14)(25)(36), (153)(264), (165432), (16)(25)(34), (1)(4)(26)(35), (12)(36)(45), (2)(5)(13)(46), (14)(23)(56), (3)(6)(15)(24)\}$

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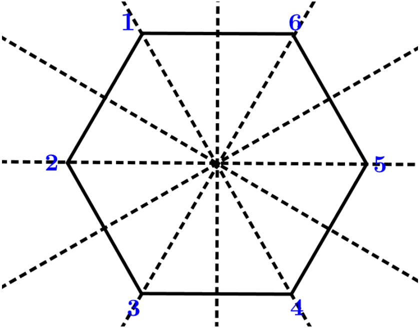

Prepare for an identification on Ducci sequences

Define $* : \mathcal{D}_6 \times A_6 \to A_6$ by

$$\pi * (a_1, a_2, \cdots, a_6) = (a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(6)})$$

for all $\pi \in \mathcal{D}_6$ and $(a_1, a_2, \cdots, a_6) \in A_6$.

Prepare for an identification on Ducci sequences

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for all $\pi \in \mathcal{D}_6$ and $(a_1, a_2, \cdots, a_6) \in A_6$.
Clearly, $*$ is well-defined.

Prepare for an identification on Ducci sequences

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for all $\pi \in \mathcal{D}_6$ and $(a_1, a_2, \cdots, a_6) \in A_6$.
Clearly, $*$ is well-defined.

Lemma (4.4)

$*$ is a left group action of $\mathcal{D}_6$ on A_6.

Proof

An identification on Ducci sequences

For all $\vec{x}, \vec{y} \in A_6$, define $\vec{x} \equiv \vec{y}$ by $\vec{x} = \pi * \vec{y}$ for some $\pi \in \mathcal{D}_6$.

An identification on Ducci sequences

For all $\vec{x}, \vec{y} \in A_6$, define $\vec{x} \equiv \vec{y}$ by $\vec{x} = \pi * \vec{y}$ for some $\pi \in \mathcal{D}_6$.
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An identification on Ducci sequences

For all $\vec{x}, \vec{y} \in A_6$, define $\vec{x} \equiv \vec{y}$ by $\vec{x} = \pi * \vec{y}$ for some $\pi \in \mathcal{D}_6$. Then, $\equiv$ is the equivalence relation on A_6 induced by $\mathcal{D}_6$ and we denote an equivalence class of A_6 by $[(a_1, a_2, \cdots, a_6)]$, where $(a_1, a_2, \cdots, a_6) \in A_6$.

An identification on Ducci sequences

For all $\vec{x}, \vec{y} \in A_6$, define $\vec{x} \equiv \vec{y}$ by $\vec{x} = \pi * \vec{y}$ for some $\pi \in \mathcal{D}_6$.
Then, $\equiv$ is the equivalence relation on A_6 induced by $\mathcal{D}_6$ and we denote an equivalence class of A_6 by $[(a_1, a_2, \cdots, a_6)]$, where $(a_1, a_2, \cdots, a_6) \in A_6$.
From now on, we identify two 6-tuples $\vec{x}, \vec{y}$ in A_6, written by $\vec{x} = \vec{y}$, if and only if $\vec{x} \equiv \vec{y}$.

An identification on Ducci sequences

Remark (4.5)

In our identification, we observe that:

(a) If $\vec{a}, \vec{b} \in A_6$, then $\vec{a} = \vec{b}$ if and only if $[\,\vec{a}\,] = [\,\vec{b}\,]$.

An identification on Ducci sequences

Remark (4.5)

In our identification, we observe that:

(a) If $\vec{a}, \vec{b} \in A_6$, then $\vec{a} = \vec{b}$ if and only if $[\,\vec{a}\,] = [\,\vec{b}\,]$.

(b) According to Remark 1.3, a sequence of regular hexagons, that is, a Diffy Hexagon game, is actually a Ducci sequence of 6-tuples in A_6.

Diffy Hexagons whose components are consisting of $\{0, 1\}$

Lemma (4.9)

There are 13 *equivalence classes of* $(\mathbb{Z}_2)^6$.

Diffy Hexagons whose components are consisting of $\{0, 1\}$

Lemma (4.9)

There are 13 *equivalence classes of* $(\mathbb{Z}_2)^6$*. In fact, they are:*
$(0,0,0,0,0,0)$, $(0,0,0,1,1,1)$, $(0,0,1,0,0,1)$, $(0,0,1,0,1,1)$,
$(0,1,0,1,0,1)$, $(0,1,1,0,1,1)$, $(0,1,1,1,0,1)$, $(0,1,1,1,1,1)$,
$(1,0,0,0,0,0)$, $(1,0,0,0,0,1)$, $(1,0,0,0,1,0)$, $(1,0,0,1,1,1)$,
and $(1,1,1,1,1,1)$.

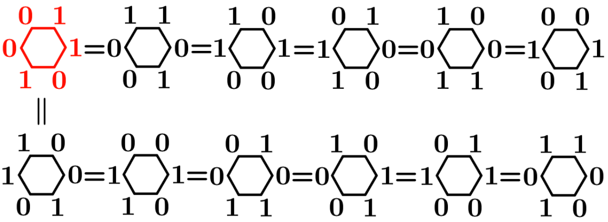
Proof

Diffy Hexagons whose components are consisting of $\{0,1\}$

Lemma (4.9)

There are 13 *equivalence classes of* $(\mathbb{Z}_2)^6$. *In fact, they are:* $(0,0,0,0,0,0)$, $(0,0,0,1,1,1)$, $(0,0,1,0,0,1)$, $(0,0,1,0,1,1)$, $(0,1,0,1,0,1)$, $(0,1,1,0,1,1)$, $(0,1,1,1,0,1)$, $(0,1,1,1,1,1)$, $(1,0,0,0,0,0)$, $(1,0,0,0,0,1)$, $(1,0,0,0,1,0)$, $(1,0,0,1,1,1)$, *and* $(1,1,1,1,1,1)$.

Proof



Cycles of Diffy Hexagons whose components are consisting of $\{0, 1\}$

Theorem (4.10)

Let $\vec{b} \in (\mathbb{Z}_2)^6$. *Then, the cycle of* $\vec{b}$ *is one of the followings:*

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$

$$\mathbf{(0, 0, 0, 0, 0, 0)}$$
$$\circlearrowleft$$

$\rightarrow$: *a Ducci process*

Cycles of Diffy Hexagons whose components are consisting of $\{0,1\}$

Theorem (4.10)

Let $\vec{b} \in (\mathbb{Z}_2)^6$. Then, the cycle of $\vec{b}$ is one of the followings:

(i) (1-cycle) $(0,0,0,0,0,0)$

$$\begin{array}{c} (0,1,0,1,0,1) \\ \downarrow \\ (1,1,1,1,1,1) \\ \downarrow \\ \mathbf{(0,0,0,0,0,0)} \\ \circlearrowleft \end{array}$$

$\rightarrow$: *a Ducci process*

Cycles of Diffy Hexagons whose components are consisting of $\{0, 1\}$

Theorem (4.10)

(ii) (1-cycle) $(0, 1, 1, 0, 1, 1)$

$$\mathbf{(0, 1, 1, 0, 1, 1)}$$

$\circlearrowleft$

$\rightarrow$: *a Ducci process*

Cycles of Diffy Hexagons whose components are consisting of $\{0, 1\}$

Theorem (4.10)

(ii) (1-cycle) $(0, 1, 1, 0, 1, 1)$

$$(0,0,0,1,1,1)$$
$$\downarrow$$
$$(0,0,1,0,0,1)$$
$$\downarrow$$
$$\mathbf{(0,1,1,0,1,1)}$$
$$\circlearrowleft$$

$\longrightarrow$: *a Ducci process*

Cycles of Diffy Hexagons whose components are consisting of $\{0, 1\}$

Theorem (4.10)

(iii) (2-cycle) $(0, 0, 1, 0, 1, 0), (0, 1, 1, 1, 1, 0)$

$$\begin{tikzcd} \mathbf{(1,0,0,0,1,0)} \arrow[r, bend left] & \mathbf{(1,0,0,1,1,1)} \arrow[l, bend left] \end{tikzcd}$$

$\rightarrow$: *a Ducci process*

Cycles of Diffy Hexagons whose components are consisting of $\{0, 1\}$

Theorem (4.10)

(iii) (2-cycle) $(0,0,1,0,1,0), (0,1,1,1,1,0)$

$$
\begin{tikzcd}
(0,1,1,1,1,1) \arrow[dr] & & (1,0,0,0,0,0) \arrow[dl] & & & (0,1,1,1,0,1) \arrow[dl] \\
& (0,1,1,1,0,1) \arrow[dr] & & & (0,1,1,1,0,1) \arrow[dl] & \\
& & \mathbf{(1,0,0,0,1,0)} \arrow[r, bend left] & \mathbf{(1,0,0,1,1,1)} \arrow[l, bend left] & &
\end{tikzcd}
$$

$\rightarrow$: *a Ducci process*

The period of Diffy Hexagons

Theorem (4.12)

Let $\vec{e}_1 = (1,0,0,0,0,0) \in A_6$. If r is a positive integer, then r is the period of $\vec{a}$ for some $\vec{a} \in A_6$ if and only if r divides the period of $\vec{e}_1$.

Proof

Thank You

Appendix

Proof for Lemma 2.1

Lemma 2.1

Proof.

Write $\vec{a} = (a_1, a_2, \cdots, a_N)$
Let $M = \max\{a_1, a_2, \cdots, a_N\}$

Proof for Lemma 2.1

Lemma 2.1

Proof.

Write $\vec{a} = (a_1, a_2, \cdots, a_N)$
Let $M = \max\{a_1, a_2, \cdots, a_N\}$
$\Longrightarrow$ There are at most $(M+1)^N$ different N-tuples which are obtained by performing Ducci processes on $\vec{a}$

Proof for Lemma 2.1

Lemma 2.1

Proof.

Write $\vec{a} = (a_1, a_2, \cdots, a_N)$
Let $M = \max\{a_1, a_2, \cdots, a_N\}$
$\implies$ There are at most $(M+1)^N$ different N-tuples which are obtained by performing Ducci processes on $\vec{a}$, and hence there are nonnegative integers n, k with $n > k$ such that $D^n(\vec{a}) = D^k(\vec{a})$ $\square$

Proof for Remark 2.4

Remark 2.4

Proof.

Note that $-M \leq x - y \leq M$

Proof for Remark 2.4

Remark 2.4

Proof.

Note that $-M \leq x - y \leq M$, then we obtain $|x - y| \leq M$ $\square$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$

If $r = s$, there is nothing to prove

Proof for Lemma 2.5

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Proof.

Given nonnegative integers r, s with $r \geq s$

If $r = s$, there is nothing to prove

Now, we may assume that $r > s$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r > s$
It suffices to show that $\max D^{s+1}(\vec{a}) \leq \max D^{s}(\vec{a})$:

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r > s$
It suffices to show that $\max D^{s+1}(\vec{a}) \leq \max D^{s}(\vec{a})$:
Write $D^{s}(\vec{a}) = (x_1, x_2, \cdots, x_N)$ and $D^{s+1}(\vec{a}) = (y_1, y_2, \cdots, y_N)$, where

$$y_1 = |x_1 - x_2|, \cdots, y_{N-1} = |x_{N-1} - x_N|, y_N = |x_N - x_1|$$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r > s$
It suffices to show that $\max D^{s+1}(\vec{a}) \leq \max D^{s}(\vec{a})$:
Write $D^{s}(\vec{a}) = (x_1, x_2, \cdots, x_N)$ and $D^{s+1}(\vec{a}) = (y_1, y_2, \cdots, y_N)$, where

$$y_1 = |x_1 - x_2|, \cdots, y_{N-1} = |x_{N-1} - x_N|, y_N = |x_N - x_1|$$

$\because 0 \leq x_1, x_2, \cdots, x_N \leq \max D^{s}(\vec{a})$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r > s$
It suffices to show that $\max D^{s+1}(\vec{a}) \leq \max D^{s}(\vec{a})$:
Write $D^{s}(\vec{a}) = (x_1, x_2, \cdots, x_N)$ and $D^{s+1}(\vec{a}) = (y_1, y_2, \cdots, y_N)$, where

$$y_1 = |x_1 - x_2|, \cdots, y_{N-1} = |x_{N-1} - x_N|, y_N = |x_N - x_1|$$

$\because 0 \leq x_1, x_2, \cdots, x_N \leq \max D^{s}(\vec{a})$
$\therefore$ By Remark 2.4, $y_i \leq \max D^{s}(\vec{a})$ for all $i = 1, 2, \cdots, N$

Proof for Lemma 2.5

Lemma 2.5

Proof.

Given nonnegative integers r, s with $r \geq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r > s$
It suffices to show that $\max D^{s+1}(\vec{a}) \leq \max D^s(\vec{a})$:
Write $D^s(\vec{a}) = (x_1, x_2, \cdots, x_N)$ and $D^{s+1}(\vec{a}) = (y_1, y_2, \cdots, y_N)$, where

$$y_1 = |x_1 - x_2|, \cdots, y_{N-1} = |x_{N-1} - x_N|, y_N = |x_N - x_1|$$

$\because 0 \leq x_1, x_2, \cdots, x_N \leq \max D^s(\vec{a})$
$\therefore$ By Remark 2.4, $y_i \leq \max D^s(\vec{a})$ for all $i = 1, 2, \cdots, N$
$\implies \max D^{s+1}(\vec{a}) \leq \max D^s(\vec{a})$ $\square$

Proof for Lemma 2.6

Lemma 2.6

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

Proof for Lemma 2.6

Lemma 2.6

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

If $r = s$, then it is trivial

Proof for Lemma 2.6

Lemma 2.6

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Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

If $r = s$, then it is trivial

Suppose $r < s$, then $\max D^s(\vec{a}) \leq \max D^r(\vec{a})$ by Lemma 2.5

Proof for Lemma 2.6

Lemma 2.6

Proof.

Given $k \leq r, s \leq n-1$
We may assume that $r \leq s$
If $r = s$, then it is trivial
Suppose $r < s$, then $\max D^s(\vec{a}) \leq \max D^r(\vec{a})$ by Lemma 2.5
Now, look at the Ducci sequence of $D^s(\vec{a})$:

$$D^s(\vec{a}), D^{s+1}(\vec{a}), \cdots, D^{n-1}(\vec{a}),$$
$$D^n(\vec{a}) = D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^r(\vec{a})$$

By Lemma 2.5, we know that $\square$

Proof for Lemma 2.6

(continued...)

$$\begin{aligned}\max D^r(\vec{a}) \leq \max D^k(\vec{a}) &= \max D^n(\vec{a}) \\ &\leq \max D^{n-1}(\vec{a}) \leq \max D^s(\vec{a})\end{aligned}$$

Proof for Lemma 2.6

(continued...)

$$\begin{aligned}\max D^r(\vec{a}) \leq \max D^k(\vec{a}) &= \max D^n(\vec{a}) \\ &\leq \max D^{n-1}(\vec{a}) \leq \max D^s(\vec{a})\end{aligned}$$

Therefore, $\max D^r(\vec{a}) \leq \max D^s(\vec{a})$

Proof for Lemma 2.6

(continued...)

$$\begin{aligned}\max D^r(\vec{a}) \leq \max D^k(\vec{a}) &= \max D^n(\vec{a}) \\ &\leq \max D^{n-1}(\vec{a}) \leq \max D^s(\vec{a})\end{aligned}$$

Therefore, $\max D^r(\vec{a}) \leq \max D^s(\vec{a})$
So, we conclude that $\max D^r(\vec{a}) = \max D^s(\vec{a})$ $\square$

Proof for Remark 2.7

Remark 2.7

Proof.

Since $|x - y| = M$, we obtain $x - y = \pm M$

Proof for Remark 2.7

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Case 1: $x-y = M$

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Since $|x - y| = M$, we obtain $x - y = \pm M$

Case 1: $x - y = M$

$\implies M + y = x \leq M$

Proof for Remark 2.7

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Since $|x - y| = M$, we obtain $x - y = \pm M$

Case 1: $x - y = M$

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$\implies y \leq 0$

Proof for Remark 2.7

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Case 1: $x - y = M$

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$\implies y \leq 0$

By assumption, $y \geq 0$

Proof for Remark 2.7

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$\therefore y = 0$

$\implies x = M$

Therefore, $x, y \in \{0, M\}$ and at least one of them is M

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Therefore, $x, y \in \{0, M\}$ and at least one of them is M

Case 2: $x - y = -M$

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Since $|x - y| = M$, we obtain $x - y = \pm M$

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By assumption, $y \geq 0$

$\therefore y = 0$

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Therefore, $x, y \in \{0, M\}$ and at least one of them is M

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Since $|x-y| = M$, we obtain $x - y = \pm M$

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By assumption, $y \geq 0$

$\therefore y = 0$

$\implies x = M$

Therefore, $x, y \in \{0, M\}$ and at least one of them is M

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Since $|x-y| = M$, we obtain $x - y = \pm M$

Case 1: $x - y = M$

$\implies M + y = x \leq M$

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By assumption, $y \geq 0$

$\therefore y = 0$

$\implies x = M$

Therefore, $x, y \in \{0, M\}$ and at least one of them is M

Case 2: $x - y = -M$

$\implies x + M = y \leq M$

$\implies x \leq 0$

Note that $x \geq 0$ □

Proof for Remark 2.7

(continued...)

$\therefore x = 0$

Proof for Remark 2.7

(continued...)

$\therefore x = 0$

$\implies y = M$

Proof for Remark 2.7

(continued...)

$\therefore\ x = 0$

$\implies y = M$

Hence, $x, y \in \{0, M\}$ and at least one of them is M □

Proof for Lemma 2.8

Lemma 2.8

Proof.

We prove it by induction on t:

Proof for Lemma 2.8

Lemma 2.8

Proof.

We prove it by induction on t:

t=1:

Proof for Lemma 2.8

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Note that $M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

Proof for Lemma 2.8

Lemma 2.8

Proof.

We prove it by induction on t:

t=1:

Note that $M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M, holds

Proof for Lemma 2.8

Lemma 2.8

Proof.

We prove it by induction on t:

t=1:

Note that $M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M, holds

Suppose $t = 1, 2, \cdots, K$ holds

Then, $t = K + 1$:

Proof for Lemma 2.8

Lemma 2.8

Proof.

We prove it by induction on t:

t=1:

Note that $M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M, holds

Suppose $t = 1, 2, \cdots, K$ holds

Then, $t = K + 1$:

By assumption, we have the following four cases:

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$

Case 4: $\exists 2 \leq i \leq K$ such that $a_i = M$ $\square$

Proof for Lemma 2.8

(continued...)

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

Proof for Lemma 2.8

(continued...)

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

$\Longrightarrow\ b_2 = b_3 = \cdots = b_K = b_{K+1} = b_{K+2}$, since $D(\vec{b}) = \vec{a}$

Proof for Lemma 2.8

(continued...)

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

$\implies b_2 = b_3 = \cdots = b_K = b_{K+1} = b_{K+2}$, since $D(\vec{b}) = \vec{a}$

$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

Proof for Lemma 2.8

(continued...)

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

$\implies b_2 = b_3 = \cdots = b_K = b_{K+1} = b_{K+2}$, since $D(\vec{b}) = \vec{a}$

$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.8

(continued...)

Case 1: $a_1 = M$ and $a_2 = a_3 = \cdots = a_K = a_{K+1} = 0$

$\implies b_2 = b_3 = \cdots = b_K = b_{K+1} = b_{K+2}$, since $D(\vec{b}) = \vec{a}$

$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M

Therefore, we obtain

$$b_1, b_2, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$$

and at least one of them is M, holds $\square$

Proof for Lemma 2.8

(continued...)

Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$

Proof for Lemma 2.8

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Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$
$\implies b_1 = b_2 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$

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(continued...)

Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$
$\implies b_1 = b_2 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \leq b_{K+1}, b_{K+2} \leq M$

Proof for Lemma 2.8

(continued...)

Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$
$\Longrightarrow\ b_1 = b_2 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \leq b_{K+1}, b_{K+2} \leq M$
$\therefore$ By Remark 2.7, we obtain $b_{K+1}, b_{K+2} \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.8

(continued...)

Case 2: $a_{K+1} = M$ and $a_1 = a_2 = a_3 = \cdots = a_K = 0$
$\implies b_1 = b_2 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \leq b_{K+1}, b_{K+2} \leq M$
$\therefore$ By Remark 2.7, we obtain $b_{K+1}, b_{K+2} \in \{0, M\}$ and at least one of them is M
$\implies b_1, b_2, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$ and at least one of them is M, holds

$\square$

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$

$\implies b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$

$\implies b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$

$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$
$\implies b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$
$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$
$\implies b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_1 = |b_1 - b_2|$ and $0 \leq b_1, b_2 \leq M$
$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M
Note that $M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \leq b_{K+1}, b_{K+2} \leq M$

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$
$\implies b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_1 = |b_1 - b_2|$ and $0 \le b_1, b_2 \le M$
$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M
Note that $M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \le b_{K+1}, b_{K+2} \le M$
By Remark 2.7, we have $b_{K+1}, b_{K+2} \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.8

(continued...)

Case 3: $a_1 = a_{K+1} = M$ and $a_2 = a_3 = \cdots = a_K = 0$
$\Longrightarrow \; b_2 = b_3 = \cdots = b_K = b_{K+1}$, since $D(\vec{b}) = \vec{a}$
$\because M = a_1 = |b_1 - b_2|$ and $0 \le b_1, b_2 \le M$
$\therefore$ By Remark 2.7, $b_1, b_2 \in \{0, M\}$ and at least one of them is M
Note that $M = a_{K+1} = |b_{K+1} - b_{K+2}|$ and $0 \le b_{K+1}, b_{K+2} \le M$
By Remark 2.7, we have $b_{K+1}, b_{K+2} \in \{0, M\}$ and at least one of them is M
So, we conclude that

$$b_1, b_2, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$$

and at least one of them is M, holds $\square$

Proof for Lemma 2.8

(continued...)

Case 4: $\exists\, 2 \leq i \leq K$ such that $a_i = M$

Proof for Lemma 2.8

(continued...)

Case 4: $\exists 2 \leq i \leq K$ such that $a_i = M$

$\because a_1, a_2, \cdots, a_i \in \{0, M\}$ and $a_i = M$ with $2 \leq i \leq K$

Proof for Lemma 2.8

(continued...)

Case 4: $\exists\, 2 \leq i \leq K$ such that $a_i = M$
$\because a_1, a_2, \cdots, a_i \in \{0, M\}$ and $a_i = M$ with $2 \leq i \leq K$
$\therefore$ By induction hypothesis, $b_1, b_2, \cdots, b_i, b_{i+1} \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.8

(continued...)

Case 4: $\exists 2 \leq i \leq K$ such that $a_i = M$

$\because a_1, a_2, \cdots, a_i \in \{0, M\}$ and $a_i = M$ with $2 \leq i \leq K$

$\therefore$ By induction hypothesis, $b_1, b_2, \cdots, b_i, b_{i+1} \in \{0, M\}$ and at least one of them is M

Note that $a_i = M, a_{i+1}, \cdots, a_{K+1} \in \{0, M\}$ and

$$\begin{aligned} 2 = (K+1) - (K-1) &\leq (K+1) - (i-1) \\ &\leq (K+1) - (2-1) \\ &= K \end{aligned}$$

$\square$

Proof for Lemma 2.8

(continued...)

Since Ducci processes are cyclic,

Proof for Lemma 2.8

(continued...)

Since Ducci processes are cyclic, we obtain

$$b_i, b_{i+1}, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$$

and at least one of them is M, by induction hypothesis

Proof for Lemma 2.8

(continued...)

Since Ducci processes are cyclic, we obtain

$$b_i, b_{i+1}, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$$

and at least one of them is M, by induction hypothesis
Hence, we conclude that

$$b_1, b_2, \cdots, b_i, b_{i+1}, \cdots, b_K, b_{K+1}, b_{K+2} \in \{0, M\}$$

and at least one of them is M, holds □

Proof for Lemma 2.10

Lemma 2.10

Proof.

We prove it by induction on i:

Proof for Lemma 2.10

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Proof.

We prove it by induction on i:

$i = 0$:

Proof for Lemma 2.10

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Proof.

We prove it by induction on i:

$i = 0$:

By Lemma 2.6, $\max D^{n-1}(\vec{a}) = \max D^{k}(\vec{a}) = M$

Proof for Lemma 2.10

Lemma 2.10

Proof.

We prove it by induction on i:

$i = 0$:

By Lemma 2.6, $\max D^{n-1}(\vec{a}) = \max D^k(\vec{a}) = M$

$\implies$ there is a component of $D^{n-1}(\vec{a})$ is M

Proof for Lemma 2.10

Lemma 2.10

Proof.

We prove it by induction on i:

$i = 0$:

By Lemma 2.6, $\max D^{n-1}(\vec{a}) = \max D^{k}(\vec{a}) = M$

$\implies$ there is a component of $D^{n-1}(\vec{a})$ is M

$\implies$ there is one cyclic consecutive component of $D^{n-1}(\vec{a})$ which is taken from 0 or M such that at least one of them is M, holds

Proof for Lemma 2.10

Lemma 2.10

Proof.

We prove it by induction on i:

$i = 0$:

By Lemma 2.6, $\max D^{n-1}(\vec{a}) = \max D^k(\vec{a}) = M$

$\implies$ there is a component of $D^{n-1}(\vec{a})$ is M

$\implies$ there is one cyclic consecutive component of $D^{n-1}(\vec{a})$ which is taken from 0

or M such that at least one of them is M, holds

Suppose $i = K$ holds

Then $i = K + 1$:

Proof for Lemma 2.10

Lemma 2.10

Proof.

We prove it by induction on i:

$i = 0$:

By Lemma 2.6, $\max D^{n-1}(\vec{a}) = \max D^k(\vec{a}) = M$

$\implies$ there is a component of $D^{n-1}(\vec{a})$ is M

$\implies$ there is one cyclic consecutive component of $D^{n-1}(\vec{a})$ which is taken from 0

or M such that at least one of them is M, holds

Suppose $i = K$ holds

Then $i = K + 1$:

We must prove that there are at least $(K + 1) + 1$ cyclic consecutive components of $D^{(n-1)-(K+1)}(\vec{a})$ taken from 0 or M such that at least one of them is M: $\square$

Proof for Lemma 2.10

(continued...)

Write

$$D^{(n-1)-(K+1)}(\vec{a}) = (x_1, \cdots, x_N) \text{ and } D^{(n-1)-K}(\vec{a}) = (y_1, \cdots, y_N)$$

Proof for Lemma 2.10

(continued...)

Write

$$D^{(n-1)-(K+1)}(\vec{a}) = (x_1, \cdots, x_N) \text{ and } D^{(n-1)-K}(\vec{a}) = (y_1, \cdots, y_N)$$

$$\implies D(x_1, x_2, \cdots, x_N) = (y_1, y_2, \cdots, y_N)$$

Proof for Lemma 2.10

(continued...)

Write

$$D^{(n-1)-(K+1)}(\vec{a}) = (x_1, \cdots, x_N) \text{ and } D^{(n-1)-K}(\vec{a}) = (y_1, \cdots, y_N)$$

$$\implies D(x_1, x_2, \cdots, x_N) = (y_1, y_2, \cdots, y_N)$$

By induction hypothesis, we know that there are at least the $K+1$ cyclic consecutive components of $D^{(n-1)-K}(\vec{a})$ are taken from 0 or M such that at least one of them is M

Proof for Lemma 2.10

(continued...)

Write

$$D^{(n-1)-(K+1)}(\vec{a}) = (x_1, \cdots, x_N) \text{ and } D^{(n-1)-K}(\vec{a}) = (y_1, \cdots, y_N)$$

$$\Longrightarrow D(x_1, x_2, \cdots, x_N) = (y_1, y_2, \cdots, y_N)$$

By induction hypothesis, we know that there are at least the $K+1$ cyclic consecutive components of $D^{(n-1)-K}(\vec{a})$ are taken from 0 or M such that at least one of them is M

Since Ducci processes are cyclic

Proof for Lemma 2.10

(continued...)

Write

$$D^{(n-1)-(K+1)}(\vec{a}) = (x_1, \cdots, x_N) \text{ and } D^{(n-1)-K}(\vec{a}) = (y_1, \cdots, y_N)$$

$$\Longrightarrow \quad D(x_1, x_2, \cdots, x_N) = (y_1, y_2, \cdots, y_N)$$

By induction hypothesis, we know that there are at least the $K+1$ cyclic consecutive components of $D^{(n-1)-K}(\vec{a})$ are taken from 0 or M such that at least one of them is M

Since Ducci processes are cyclic, we may assume $y_1, y_2, \cdots, y_K, y_{K+1}$ are $K+1$ cyclic consecutive components of $D^{(n-1)-K}(\vec{a})$ which are taken from 0 or M such that at least one of them is M without loss of generality $\square$

Proof for Lemma 2.10

(continued...)

By Lemma 2.8, $x_1, x_2, \cdots, x_K, x_{K+1}, x_{K+2} \in \{0, M\}$ and at least one of them is M

Proof for Lemma 2.10

(continued...)

By Lemma 2.8, $x_1, x_2, \cdots, x_K, x_{K+1}, x_{K+2} \in \{0, M\}$ and at least one of them is M
Hence, we conclude that $x_1, x_2, \cdots, x_K, x_{K+1}, x_{K+2}$ are $(K+1)+1$ cyclic consecutive components of $D^{(n-1)-(K+1)}(\vec{a})$ which are taken from 0 or M such that at least one of them is M, so $i = K+1$ holds $\square$

Proof for Remark 2.11

Remark 2.11

Proof.

The first statement follows from the fact that A_N is a collection of N-tuples of nonnegative integers
Now, we prove the last statement:

Proof for Remark 2.11

Remark 2.11

Proof.

The first statement follows from the fact that A_N is a collection of N-tuples of nonnegative integers

Now, we prove the last statement:

$\because i \leq \min\{n-k-1, N-1\}$

Proof for Remark 2.11

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Proof.

The first statement follows from the fact that A_N is a collection of N-tuples of nonnegative integers

Now, we prove the last statement:

$\because i \leq \min\{n-k-1, N-1\}$

$\therefore i \leq n-k-1$

Proof for Remark 2.11

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Proof.

The first statement follows from the fact that A_N is a collection of N-tuples of nonnegative integers

Now, we prove the last statement:

$\because\ i \leq \min\{n-k-1, N-1\}$

$\therefore\ i \leq n-k-1$

By **(a)**, we have $0 \leq i \leq n-k-1$

Proof for Remark 2.11

Remark 2.11

Proof.

The first statement follows from the fact that A_N is a collection of N-tuples of nonnegative integers

Now, we prove the last statement:

$\because\ i \leq \min\{n-k-1, N-1\}$

$\therefore\ i \leq n-k-1$

By **(a)**, we have $0 \leq i \leq n-k-1$ and

$$\begin{aligned} k = (n-1) - (n-k-1) &\leq (n-1) - i \\ &\leq (n-1) - 0 \\ &= n-1 \end{aligned}$$

$\square$

Proof for Theorem 2.12

Theorem 2.12

Proof.

Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle

Proof for Theorem 2.12

Theorem 2.12

Proof.

Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle
By Lemma 2.6, we obtain $\max D^j(\vec{a}) = \max D^k(\vec{a}) = M$ for all $j = k, k+1, \cdots, n-1$

Proof for Theorem 2.12

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Proof.

Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle
By Lemma 2.6, we obtain $\max D^j(\vec{a}) = \max D^k(\vec{a}) = M$ for all $j = k, k+1, \cdots, n-1$
In particular, $D^{n-1}(\vec{a}) = M$

Proof for Theorem 2.12

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Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle
By Lemma 2.6, we obtain $\max D^j(\vec{a}) = \max D^k(\vec{a}) = M$ for all $j = k, k+1, \cdots, n-1$
In particular, $D^{n-1}(\vec{a}) = M$
By Lemma 2.10, we know that there are at least N cyclic consecutive components of $D^{(n-1)-(N-1)}(\vec{a})$ taken from 0 or M

Proof for Theorem 2.12

Theorem 2.12

Proof.

Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle
By Lemma 2.6, we obtain $\max D^j(\vec{a}) = \max D^k(\vec{a}) = M$ for all $j = k, k+1, \cdots, n-1$
In particular, $D^{n-1}(\vec{a}) = M$
By Lemma 2.10, we know that there are at least N cyclic consecutive components of $D^{(n-1)-(N-1)}(\vec{a})$ taken from 0 or M
$\Longrightarrow$ the components of $D^{n-N}(\vec{a})$ are all equal to either 0 or M which follows from $D^{n-N}(\vec{a}) \in A_N$

Proof for Theorem 2.12

Theorem 2.12

Proof.

Therefore, $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle
By Lemma 2.6, we obtain $\max D^j(\vec{a}) = \max D^k(\vec{a}) = M$ for all $j = k, k+1, \cdots, n-1$
In particular, $D^{n-1}(\vec{a}) = M$
By Lemma 2.10, we know that there are at least N cyclic consecutive components of $D^{(n-1)-(N-1)}(\vec{a})$ taken from 0 or M
$\implies$ the components of $D^{n-N}(\vec{a})$ are all equal to either 0 or M which follows from $D^{n-N}(\vec{a}) \in A_N$
Now, look at the Ducci sequence of $D^{n-N}(\vec{a})$:

□

Proof for Theorem 2.12

(continued...)

$$D^{n-N}(\vec{a}), D^{n-N+1}(\vec{a}), \cdots, D^{n-1}(\vec{a}), D^{n}(\vec{a}) = D^{k}(\vec{a}),$$
$$D^{k+1}(\vec{a}), D^{k+2}(\vec{a}), \cdots, D^{n-N-1}(\vec{a}), \cdots$$

Proof for Theorem 2.12

(continued...)

$$D^{n-N}(\vec{a}), D^{n-N+1}(\vec{a}), \cdots, D^{n-1}(\vec{a}), D^{n}(\vec{a}) = D^{k}(\vec{a}),$$
$$D^{k+1}(\vec{a}), D^{k+2}(\vec{a}), \cdots, D^{n-N-1}(\vec{a}), \cdots$$

$\Longrightarrow$ the components of $D^{k}(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all equal to either 0 or M, since the components of $D^{n-N}(\vec{a})$ are all equal to 0 or M

Hence, we complete this proof $\square$

Proof for Remark 2.13

Remark 2.13

Proof.

Let $M > 1$ be an integer

Proof for Remark 2.13

Remark 2.13

Proof.

Let $M > 1$ be an integer and $\vec{a} = (M, 0, 1, \cdots, 1, 1) \in A_N$

Proof for Remark 2.13

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Proof.

Let $M > 1$ be an integer and $\vec{a} = (M, 0, 1, \cdots, 1, 1) \in A_N$
Choose $\vec{b} = D(\vec{a}) \in A_N$

Proof for Remark 2.13

Remark 2.13

Proof.

Let $M > 1$ be an integer and $\vec{a} = (M, 0, 1, \cdots, 1, 1) \in A_N$
Choose $\vec{b} = D(\vec{a}) \in A_N$
$\implies \vec{b} = (M, 1, 0, \cdots, 0, M - 1)$

Proof for Remark 2.13

Remark 2.13

Proof.

Let $M > 1$ be an integer and $\vec{a} = (M, 0, 1, \cdots, 1, 1) \in A_N$
Choose $\vec{b} = D(\vec{a}) \in A_N$
$\implies \vec{b} = (M, 1, 0, \cdots, 0, M-1)$
Note that $\max \vec{a} = \max \vec{b} = M$

Proof for Remark 2.13

Remark 2.13

Proof.

Let $M > 1$ be an integer and $\vec{a} = (M, 0, 1, \cdots, 1, 1) \in A_N$
Choose $\vec{b} = D(\vec{a}) \in A_N$
$\Longrightarrow \vec{b} = (M, 1, 0, \cdots, 0, M-1)$
Note that $\max \vec{a} = \max \vec{b} = M$
$\therefore$ the components of $\vec{a}$, $\vec{b}$ aren't all equal to either 0 or M $\square$

Proof for Lemma 2.15

Lemma 2.15

Proof.

Let $\gcd \vec{a} = d$

Write $\vec{a} = d\vec{b}$ with $\gcd \vec{b} = 1$

Proof for Lemma 2.15

Lemma 2.15

Proof.

Let $\gcd \vec{a} = d$
Write $\vec{a} = d\vec{b}$ with $\gcd \vec{b} = 1$
Note that $d > 0$ and $D(\vec{a}) = D(d\vec{b}) = dD(\vec{b})$

Proof for Lemma 2.15

Lemma 2.15

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Let $\gcd \vec{a} = d$

Write $\vec{a} = d\vec{b}$ with $\gcd \vec{b} = 1$

Note that $d > 0$ and $D(\vec{a}) = D(d\vec{b}) = dD(\vec{b})$

$\implies D^n(\vec{a}) = D^n(d\vec{b}) = dD^n(\vec{b})$ by induction on n

Proof for Lemma 2.15

Lemma 2.15

Proof.

Let $\gcd \vec{a} = d$

Write $\vec{a} = d\vec{b}$ with $\gcd \vec{b} = 1$

Note that $d > 0$ and $D(\vec{a}) = D(d\vec{b}) = dD(\vec{b})$

$\implies D^n(\vec{a}) = D^n(d\vec{b}) = dD^n(\vec{b})$ by induction on n

$\implies d \mid D^n(\vec{a})$

Therefore, we know that $\gcd \vec{a} \mid D^n(\vec{a})$ $\square$

Proof for Corollary 2.16

Corollary 2.16

Proof.

It follows from Theorem 2.12 and Lemma 2.15 □

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

Proof for Lemma 2.18

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It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$
Given nonnegative integers r, s with $r \leq s$

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Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

Given nonnegative integers r, s with $r \leq s$

If $r = s$, there is nothing to prove

Now, we may assume that $r < s$

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

Given nonnegative integers r, s with $r \leq s$

If $r = s$, there is nothing to prove

Now, we may assume that $r < s$

It is reduced to prove that $\gcd D^r(\vec{a}) \mid \gcd D^{r+1}(\vec{a})$:

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$
Given nonnegative integers r, s with $r \leq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r < s$
It is reduced to prove that $\gcd D^r(\vec{a}) \mid \gcd D^{r+1}(\vec{a})$:
Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that $\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$

Proof for Lemma 2.18

Lemma 2.18

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It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$
Given nonnegative integers r, s with $r \leq s$
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Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that
$\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$
$\implies d > 0$, since $D^r(\vec{a}) \in A_N$

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

Given nonnegative integers r, s with $r \le s$

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Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that

$\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$

$\implies d > 0$, since $D^r(\vec{a}) \in A_N$

$\implies D^{r+1}(\vec{a}) = (\,|x_1 - x_2|d, \cdots, |x_{N-1} - x_N|d, |x_N - x_1|d\,)$

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

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Now, we may assume that $r < s$

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Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that

$\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$

$\implies d > 0$, since $D^r(\vec{a}) \in A_N$

$\implies D^{r+1}(\vec{a}) = (\,|x_1 - x_2|d, \cdots, |x_{N-1} - x_N|d, |x_N - x_1|d\,)$

Let $\gcd(\,|x_1 - x_2|, \cdots, |x_{N-1} - x_N|, |x_N - x_1|\,) = d^*$

Proof for Lemma 2.18

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Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$
Given nonnegative integers r, s with $r \leq s$
If $r = s$, there is nothing to prove
Now, we may assume that $r < s$
It is reduced to prove that $\gcd D^r(\vec{a}) \mid \gcd D^{r+1}(\vec{a})$:
Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that
$\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$
$\implies d > 0$, since $D^r(\vec{a}) \in A_N$
$\implies D^{r+1}(\vec{a}) = (\,|x_1 - x_2|d, \cdots, |x_{N-1} - x_N|d, |x_N - x_1|\,d\,)$
Let $\gcd(\,|x_1 - x_2|, \cdots, |x_{N-1} - x_N|, |x_N - x_1|\,) = d^*$
$\implies \gcd D^{r+1}(\vec{a}) = d^* \cdot d = d^* \cdot \gcd D^r(\vec{a})$

Proof for Lemma 2.18

Lemma 2.18

Proof.

It suffices to show that $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$

Given nonnegative integers r, s with $r \leq s$

If $r = s$, there is nothing to prove

Now, we may assume that $r < s$

It is reduced to prove that $\gcd D^r(\vec{a}) \mid \gcd D^{r+1}(\vec{a})$:

Write $D^r(\vec{a}) = (x_1 d, x_2 d, \cdots, x_N d)$ such that

$\gcd(x_1, x_2, \cdots, x_N) = 1$, where $\gcd D^r(\vec{a}) = d$

$\implies d > 0$, since $D^r(\vec{a}) \in A_N$

$\implies D^{r+1}(\vec{a}) = (\,|x_1 - x_2|d, \cdots, |x_{N-1} - x_N|d, |x_N - x_1|d\,)$

Let $\gcd(\,|x_1 - x_2|, \cdots, |x_{N-1} - x_N|, |x_N - x_1|\,) = d^*$

$\implies \gcd D^{r+1}(\vec{a}) = d^* \cdot d = d^* \cdot \gcd D^r(\vec{a})$

$\therefore \gcd D^r(\vec{a}) \mid \gcd D^{r+1}(\vec{a})$ $\square$

Proof for Lemma 2.19

Lemma 2.19

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

Proof for Lemma 2.19

Lemma 2.19

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

If $r = s$, then it is trivial

Proof for Lemma 2.19

Lemma 2.19

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

If $r = s$, then it is trivial

Suppose $r < s$, then $\gcd D^r(\vec{a}) \leq \gcd D^s(\vec{a})$ by Lemma 2.18

Proof for Lemma 2.19

Lemma 2.19

Proof.

Given $k \leq r, s \leq n-1$
We may assume that $r \leq s$
If $r = s$, then it is trivial
Suppose $r < s$, then $\gcd D^r(\vec{a}) \leq \gcd D^s(\vec{a})$ by Lemma 2.18
Now, look at the Ducci sequence of $D^s(\vec{a})$:

$$D^s(\vec{a}), D^{s+1}(\vec{a}), \cdots, D^{n-1}(\vec{a}),$$
$$D^n(\vec{a}) = D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^r(\vec{a})$$

Proof for Lemma 2.19

Lemma 2.19

Proof.

Given $k \leq r, s \leq n-1$

We may assume that $r \leq s$

If $r = s$, then it is trivial

Suppose $r < s$, then $\gcd D^r(\vec{a}) \leq \gcd D^s(\vec{a})$ by Lemma 2.18

Now, look at the Ducci sequence of $D^s(\vec{a})$:

$$D^s(\vec{a}), D^{s+1}(\vec{a}), \cdots, D^{n-1}(\vec{a}),$$
$$D^n(\vec{a}) = D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^r(\vec{a})$$

By Lemma 2.18, we know that □

Proof for Lemma 2.19

(continued...)

$$\gcd D^s(\vec{a}) \leq \gcd D^k(\vec{a}) = \gcd D^n(\vec{a}) \leq \gcd D^{k+1}(\vec{a}) \leq \gcd D^r(\vec{a})$$

Proof for Lemma 2.19

(continued...)

$$\gcd D^s(\vec{a}) \leq \gcd D^k(\vec{a}) = \gcd D^n(\vec{a}) \leq \gcd D^{k+1}(\vec{a}) \leq \gcd D^r(\vec{a})$$

Therefore, $\gcd D^s(\vec{a}) \leq \gcd D^r(\vec{a})$

Proof for Lemma 2.19

(continued...)

$$\gcd D^s(\vec{a}) \leq \gcd D^k(\vec{a}) = \gcd D^n(\vec{a}) \leq \gcd D^{k+1}(\vec{a}) \leq \gcd D^r(\vec{a})$$

Therefore, $\gcd D^s(\vec{a}) \leq \gcd D^r(\vec{a})$
So, we conclude that $\gcd D^r(\vec{a}) = \gcd D^s(\vec{a})$ $\square$

Proof for Theorem 3.2

Theorem 3.2

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$ and

$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$-cycle of $\vec{a}$

Proof for Theorem 3.2

Theorem 3.2

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$ and

$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$-cycle of $\vec{a}$
Let $D^k(\vec{a}) = (x_1, x_2, \cdots, x_N)$

Proof for Theorem 3.2

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$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$-cycle of $\vec{a}$
Let $D^k(\vec{a}) = (x_1, x_2, \cdots, x_N)$
By Theorem 2.12, $x_1, x_2, \cdots, x_N \in \{0, M\}$, where $M = \max D^k(\vec{a})$

Proof for Theorem 3.2

Theorem 3.2

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$ and

$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$-cycle of $\vec{a}$
Let $D^k(\vec{a}) = (x_1, x_2, \cdots, x_N)$
By Theorem 2.12, $x_1, x_2, \cdots, x_N \in \{0, M\}$, where $M = \max D^k(\vec{a})$
Since $D^k(\vec{a}) \in A_N$, we know that M is a nonnegative integer

Proof for Theorem 3.2

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By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$ and

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Let $D^k(\vec{a}) = (x_1, x_2, \cdots, x_N)$
By Theorem 2.12, $x_1, x_2, \cdots, x_N \in \{0, M\}$, where $M = \max D^k(\vec{a})$
Since $D^k(\vec{a}) \in A_N$, we know that M is a nonnegative integer
$\Longrightarrow$ We have the following two cases:
Case 1: $M = 0$
Case 2: $M > 0$ $\square$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

$\therefore k \leq n - 1$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

$\therefore k \leq n - 1$

Therefore, we have $n - 1 = k$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

$\therefore k \leq n - 1$

Therefore, we have $n - 1 = k$

So, the period of $\vec{a}$ is $n - k = 1$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

$\therefore k \leq n - 1$

Therefore, we have $n - 1 = k$

So, the period of $\vec{a}$ is $n - k = 1$

Choose $\vec{b} = \vec{\mathbf{0}} \in (\mathbb{Z}_2)^N$

Proof for Theorem 3.2

(continued...)

Case 1: $M = 0$

$\implies D^k(\vec{a}) = (0, 0, \cdots, 0) \in (\mathbb{Z}_2)^N$

$\implies$

$D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = D^k(\vec{a})$

$\implies n - 1 \leq k$, since $\vec{a}, D(\vec{a}), \cdots, D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\because$ the period of $\vec{a}$ is $n - k$

$\therefore k \leq n - 1$

Therefore, we have $n - 1 = k$

So, the period of $\vec{a}$ is $n - k = 1$

Choose $\vec{b} = \vec{\mathbf{0}} \in (\mathbb{Z}_2)^N$

$\implies D(\vec{b}) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = \vec{b} = D^0(\vec{b})$

$\square$

Proof for Theorem 3.2

(continued...)

$\Longrightarrow$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Proof for Theorem 3.2

(continued...)

$\Longrightarrow$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Proof for Theorem 3.2

(continued...)

$\Longrightarrow$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a})=\vec{\mathbf{0}}=D^0(\vec{b})$

Proof for Theorem 3.2

(continued...)

$\implies$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a})=\vec{\mathbf{0}}=D^0(\vec{b})$

$\implies$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Proof for Theorem 3.2

(continued...)

$\implies$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a})=\vec{\mathbf{0}}=D^0(\vec{b})$

$\implies$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Case 2: $M>0$

Proof for Theorem 3.2

(continued...)

$\implies$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b}) = \vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a}) = \vec{\mathbf{0}} = D^0(\vec{b})$

$\implies$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Case 2: $M > 0$

$\implies M \in \mathbb{N}$

Proof for Theorem 3.2

(continued...)

$\Longrightarrow$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a})=\vec{\mathbf{0}}=D^0(\vec{b})$

$\Longrightarrow$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Case 2: $M>0$

$\Longrightarrow$ $M\in\mathbb{N}$

Write $D^k(\vec{a})=(x_1,x_2,\cdots,x_N)=M(y_1,y_2,\cdots,y_N)$, where $y_1,y_2,\cdots,y_N$ are taken from 0 or 1

Proof for Theorem 3.2

(continued...)

$\implies$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b}) = \vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a}) = \vec{\mathbf{0}} = D^0(\vec{b})$

$\implies$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Case 2: $M > 0$

$\implies M \in \mathbb{N}$

Write $D^k(\vec{a}) = (x_1, x_2, \cdots, x_N) = M(y_1, y_2, \cdots, y_N)$, where $y_1, y_2, \cdots, y_N$ are taken from 0 or 1

Choose $\vec{b} = (y_1, y_2, \cdots, y_N) \in (\mathbb{Z}_2)^N$

Proof for Theorem 3.2

(continued...)

$\implies$ the period of $\vec{b}$ is $(1-0)=1$ and the 1-cycle of $\vec{b}$ is $D^0(\vec{b})=\vec{\mathbf{0}}$

Therefore, the period of $\vec{a}$ is equal to the period of $\vec{b}$

Note that $D^{k+1}(\vec{a})=\vec{\mathbf{0}}=D^0(\vec{b})$

$\implies$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we are done

Case 2: $M>0$

$\implies M\in\mathbb{N}$

Write $D^k(\vec{a})=(x_1,x_2,\cdots,x_N)=M(y_1,y_2,\cdots,y_N)$, where $y_1,y_2,\cdots,y_N$ are taken from 0 or 1

Choose $\vec{b}=(y_1,y_2,\cdots,y_N)\in(\mathbb{Z}_2)^N$

$\implies D^k(\vec{a})=M\vec{b}$ $\square$

Proof for Theorem 3.2

(continued...)

$$\Longrightarrow \; D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$$

Proof for Theorem 3.2

(continued...)

$$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$$
$$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$$

Proof for Theorem 3.2

(continued...)

$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$
$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$

In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$

Proof for Theorem 3.2

(continued...)

$\Longrightarrow \; D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$

$\Longrightarrow \; D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$

In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$

$\Longrightarrow \; D^0(\vec{b}) = \vec{b} = D^{n-k}(\vec{b})$

Proof for Theorem 3.2

(continued...)

$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$
$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$
In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$
$\implies D^0(\vec{b}) = \vec{b} = D^{n-k}(\vec{b})$
By assumption,
$D^k(\vec{a}) = M\vec{b}, D^{k+1}(\vec{a}) = MD(\vec{b}), \cdots, D^{n-1}(\vec{a}) = D^{n-k-1}(\vec{b})$ are all distinct

Proof for Theorem 3.2

(continued...)

$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$

$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$

In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$

$\implies D^0(\vec{b}) = \vec{b} = D^{n-k}(\vec{b})$

By assumption,

$D^k(\vec{a}) = M\vec{b}, D^{k+1}(\vec{a}) = MD(\vec{b}), \cdots, D^{n-1}(\vec{a}) = D^{n-k-1}(\vec{b})$ are all distinct

$\implies D^0(\vec{b}) = \vec{b}, D(\vec{b}), \cdots, D^{n-k-1}(\vec{b})$ are all distinct

Proof for Theorem 3.2

(continued...)

$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$

$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$

In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$

$\implies D^0(\vec{b}) = \vec{b} = D^{n-k}(\vec{b})$

By assumption,
$D^k(\vec{a}) = M\vec{b}, D^{k+1}(\vec{a}) = MD(\vec{b}), \cdots, D^{n-1}(\vec{a}) = D^{n-k-1}(\vec{b})$ are all distinct

$\implies D^0(\vec{b}) = \vec{b}, D(\vec{b}), \cdots, D^{n-k-1}(\vec{b})$ are all distinct

Therefore, the period of $\vec{b}$ is $(n-k) - 0 = n-k$ and the $(n-k)$-cycle of $\vec{b}$ is $D^0(\vec{b}) = \vec{b}, D(\vec{b}), \cdots, D^{n-k-1}(\vec{b})$

Proof for Theorem 3.2

(continued...)

$\implies D^{k+1}(\vec{a}) = D(D^k(\vec{a})) = D(M\vec{b}) = MD(\vec{b})$

$\implies D^{k+i}(\vec{a}) = MD^i(\vec{b}), \forall\, i = 1, 2, \cdots, n-k$

In particular, $M\vec{b} = D^k(\vec{a}) = D^n(\vec{a}) = MD^{n-k}(\vec{b})$

$\implies D^0(\vec{b}) = \vec{b} = D^{n-k}(\vec{b})$

By assumption,

$D^k(\vec{a}) = M\vec{b}, D^{k+1}(\vec{a}) = MD(\vec{b}), \cdots, D^{n-1}(\vec{a}) = D^{n-k-1}(\vec{b})$ are all distinct

$\implies D^0(\vec{b}) = \vec{b}, D(\vec{b}), \cdots, D^{n-k-1}(\vec{b})$ are all distinct

Therefore, the period of $\vec{b}$ is $(n-k) - 0 = n-k$ and the $(n-k)$-cycle of $\vec{b}$ is $D^0(\vec{b}) = \vec{b}, D(\vec{b}), \cdots, D^{n-k-1}(\vec{b})$

So, the period of $\vec{a}$ is equal to the period of $\vec{b}$

$\square$

Proof for Theorem 3.2

(continued...)

$\because D^k(\vec{a}) = M\vec{b} = MD^0(\vec{b})$

Proof for Theorem 3.2

(continued...)

$\because D^k(\vec{a}) = M\vec{b} = MD^0(\vec{b})$

$\therefore$ the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Hence, we complete this proof $\square$

Proof for Lemma 3.4

Lemma 3.4

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

Proof for Lemma 3.4

Lemma 3.4

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$
Since the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$, $\exists\, m \in \mathbb{N}$ such that

$$D^r(\vec{a}) = mD^s(\vec{b}),$$

where r, s are nonnegative integers with $k \le s \le n - 1$

Proof for Lemma 3.4

Lemma 3.4

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$
Since the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$, $\exists\, m \in \mathbb{N}$ such that

$$D^r(\vec{a}) = mD^s(\vec{b}),$$

where r, s are nonnegative integers with $k \leq s \leq n-1$
Then, we have:

$$\begin{aligned} D^{r+1}(\vec{a^c}) &= D^r(D(\vec{a^c})) \\ &= D^r(D(\vec{a})) \\ &= D^{r+1}(\vec{a}) \end{aligned}$$

□

Proof for Lemma 3.4

Proof.

$$
\begin{aligned}
&= D(D^r(\vec{a})) \\
&= D(mD^s(\vec{b})) \\
&= mD(D^s(\vec{b})) \\
&= mD^{s+1}(\vec{b})
\end{aligned}
$$

$\Longrightarrow D^{r+1}(\vec{a^c}) = mD^{s+1}(\vec{b})$ which completes this proof $\square$

Proof for Remark 3.5

Remark 3.5

Proof.

If $k \leq s < n-1$, then it is trivial

Now, we may assume that $s = n-1$:

Proof for Remark 3.5

Remark 3.5

Proof.

If $k \leq s < n-1$, then it is trivial

Now, we may assume that $s = n-1$:

$\implies s+1 = n$

Proof for Remark 3.5

Remark 3.5

Proof.

If $k \leq s < n-1$, then it is trivial

Now, we may assume that $s = n-1$:

$\implies s+1 = n$

$\implies D^{s+1}(\vec{b}) = D^{n}(\vec{b}) = D^{k}(\vec{b})$ is in the cycle of $\vec{b}$ $\square$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$\Longrightarrow \exists\, k \leq r \leq n-1$ such that $\vec{a^c} = D^r(\vec{a})$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$$\Longrightarrow \exists\, k \le r \le n-1 \text{ such that } \vec{a^c} = D^r(\vec{a})$$

$$\Longrightarrow D(\vec{a}) = D(\vec{a^c}) = D(D^r(\vec{a})) = D^{r+1}(\vec{a})$$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$$\begin{aligned}
&\Longrightarrow \exists\, k \le r \le n-1 \text{ such that } \vec{a^c} = D^r(\vec{a}) \\
&\Longrightarrow D(\vec{a}) = D(\vec{a^c}) = D(D^r(\vec{a})) = D^{r+1}(\vec{a}) \\
&\Longrightarrow n-1 \le r, \text{ since } D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, \\
&D^k(\vec{a}), \cdots, D^r(\vec{a}), \cdots, D^{n-1}(\vec{a}) \text{ are all distinct}
\end{aligned}$$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$\Longrightarrow$ $\exists\, k \leq r \leq n-1$ such that $\vec{a^c} = D^r(\vec{a})$

$\Longrightarrow$ $D(\vec{a}) = D(\vec{a^c}) = D(D^r(\vec{a})) = D^{r+1}(\vec{a})$

$\Longrightarrow$ $n-1 \leq r$, since $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots,$
$D^k(\vec{a}), \cdots, D^r(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

Therefore, $r = n-1$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$\Longrightarrow \exists\, k \leq r \leq n-1$ such that $\vec{a^c} = D^r(\vec{a})$

$\Longrightarrow D(\vec{a}) = D(\vec{a^c}) = D(D^r(\vec{a})) = D^{r+1}(\vec{a})$

$\Longrightarrow n-1 \leq r$, since $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots,$
$D^k(\vec{a}), \cdots, D^r(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

Therefore, $r = n-1$

$\Longrightarrow n = r+1$

Proof for Lemma 3.6

Lemma 3.6

Proof.

Note that $D(\vec{a^c}) = (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_N - a_1|) = D(\vec{a})$

"$\Rightarrow$" Suppose the condition holds

$\Longrightarrow \ \exists\, k \le r \le n-1$ such that $\vec{a^c} = D^r(\vec{a})$

$\Longrightarrow \ D(\vec{a}) = D(\vec{a^c}) = D(D^r(\vec{a})) = D^{r+1}(\vec{a})$

$\Longrightarrow \ n-1 \le r$, since $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots,$ $D^k(\vec{a}), \cdots, D^r(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

Therefore, $r = n-1$

$\Longrightarrow \ n = r+1$

Then, we have:

$$D^k(\vec{a}) = D^n(\vec{a}) = D^{r+1}(\vec{a}) = D(\vec{a^c}) = D(\vec{a}) \qquad (*)$$

$\square$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$
By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$,
$D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$

By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$, $D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\implies n - 1 \leq k - 1$, by $(*)$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$

By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$, $D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\Longrightarrow \ n - 1 \leq k - 1$, by $(*)$

$\Longrightarrow \ n \leq k$ which is a contradiction to $n > k$ $\square$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$

By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$, $D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\Longrightarrow \; n-1 \leq k-1$, by $(*)$

$\Longrightarrow \; n \leq k$ which is a contradiction to $n > k$ $\square$

By **Claim** and $(*)$, we obtain $\vec{a} = D^0(\vec{a}) = D(\vec{a^c})$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$
By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$, $D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct
$\Longrightarrow$ $n-1 \leq k-1$, by $(*)$
$\Longrightarrow$ $n \leq k$ which is a contradiction to $n > k$ $\square$

By **Claim** and $(*)$, we obtain $\vec{a} = D^0(\vec{a}) = D(\vec{a^c})$
$\Longrightarrow$ $\forall 1 \leq i \leq N, a_i = M - a_i$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" **Claim:** $k = 0$

Proof.

If not, suppose $k \geq 1$

By assumption, $D^0(\vec{a}) = \vec{a}, D(\vec{a}), \cdots, D^{k-1}(\vec{a})$, $D^k(\vec{a}), \cdots, D^{n-1}(\vec{a})$ are all distinct

$\implies n - 1 \leq k - 1$, by $(*)$

$\implies n \leq k$ which is a contradiction to $n > k$ $\square$

By **Claim** and $(*)$, we obtain $\vec{a} = D^0(\vec{a}) = D(\vec{a^c})$

$\implies \forall 1 \leq i \leq N, a_i = M - a_i$

$\implies \forall 1 \leq i \leq N, a_i = \dfrac{M}{2}$

$\square$

Proof for Lemma 3.6

(continued...)

$$\text{“}\Rightarrow\text{”} \quad \Longrightarrow \; M = \max \vec{a} = \frac{M}{2}$$

Proof for Lemma 3.6

(continued...)

$$\text{“}\Rightarrow\text{”} \quad \Longrightarrow \quad M = \max \vec{a} = \frac{M}{2}$$
$$\Longrightarrow \quad M = 0$$

Proof for Lemma 3.6

(continued...)

$$\text{“}\Rightarrow\text{”} \quad \Longrightarrow \quad M = \max \vec{a} = \frac{M}{2}$$

$$\Longrightarrow \quad M = 0$$

$$\because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0$$

Proof for Lemma 3.6

(continued...)

$$
\begin{aligned}
\text{“}\Rightarrow\text{”} \quad & \Longrightarrow \ M = \max \vec{a} = \frac{M}{2} \\
& \Longrightarrow \ M = 0 \\
& \because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0 \\
& \therefore a_1 = a_2 = \cdots = a_N = 0
\end{aligned}
$$

Proof for Lemma 3.6

(continued...)

$$
\begin{aligned}
\text{“}\Rightarrow\text{”} \quad & \Longrightarrow \quad M = \max \vec{a} = \frac{M}{2} \\
& \Longrightarrow \quad M = 0 \\
& \because 0 \le a_1, a_2, \cdots, a_N \le M = 0 \\
& \therefore a_1 = a_2 = \cdots = a_N = 0 \\
& \Longrightarrow \quad \vec{a} = \vec{\mathbf{0}}
\end{aligned}
$$

Proof for Lemma 3.6

(continued...)

$$
\begin{aligned}
\text{“}\Rightarrow\text{”} \quad & \implies M = \max \vec{a} = \frac{M}{2} \\
& \implies M = 0 \\
& \because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0 \\
& \therefore a_1 = a_2 = \cdots = a_N = 0 \\
& \implies \vec{a} = \vec{\mathbf{0}}
\end{aligned}
$$

“$\Leftarrow$” Suppose $\vec{a} = \vec{\mathbf{0}}$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" $\implies M = \max \vec{a} = \dfrac{M}{2}$

$\implies M = 0$

$\because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0$

$\therefore a_1 = a_2 = \cdots = a_N = 0$

$\implies \vec{a} = \vec{\mathbf{0}}$

"$\Leftarrow$" Suppose $\vec{a} = \vec{\mathbf{0}}$

$\implies a_1 = a_2 = \cdots = a_N = 0$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" $\Longrightarrow M = \max \vec{a} = \dfrac{M}{2}$

$\Longrightarrow M = 0$

$\because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0$

$\therefore a_1 = a_2 = \cdots = a_N = 0$

$\Longrightarrow \vec{a} = \vec{\mathbf{0}}$

"$\Leftarrow$" Suppose $\vec{a} = \vec{\mathbf{0}}$

$\Longrightarrow a_1 = a_2 = \cdots = a_N = 0$

$\Longrightarrow M = 0$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" $\Longrightarrow\ M = \max \vec{a} = \dfrac{M}{2}$

$\Longrightarrow\ M = 0$

$\because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0$

$\therefore a_1 = a_2 = \cdots = a_N = 0$

$\Longrightarrow\ \vec{a} = \vec{\mathbf{0}}$

"$\Leftarrow$" Suppose $\vec{a} = \vec{\mathbf{0}}$

$\Longrightarrow\ a_1 = a_2 = \cdots = a_N = 0$

$\Longrightarrow\ M = 0$

$\Longrightarrow\ \vec{a^c} = (0, 0, \cdots, 0)$

Proof for Lemma 3.6

(continued...)

"$\Rightarrow$" $\implies M = \max \vec{a} = \dfrac{M}{2}$

$\implies M = 0$

$\because 0 \leq a_1, a_2, \cdots, a_N \leq M = 0$

$\therefore a_1 = a_2 = \cdots = a_N = 0$

$\implies \vec{a} = \vec{\mathbf{0}}$

"$\Leftarrow$" Suppose $\vec{a} = \vec{\mathbf{0}}$

$\implies a_1 = a_2 = \cdots = a_N = 0$

$\implies M = 0$

$\implies \vec{a^c} = (0, 0, \cdots, 0)$

Note that

$D(\vec{a}) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = \vec{a} = D^0(\vec{a})$

□

Proof for Lemma 3.6

(continued...)

"$\Leftarrow$" $\Longrightarrow$ the period of $\vec{a}$ is $(1-0)=1$

Proof for Lemma 3.6

(continued...)

"$\Leftarrow$" $\Longrightarrow$ the period of $\vec{a}$ is $(1-0)=1$ and the 1-cycle of $\vec{a}$ is

$$D^0(\vec{a}) = (0,0,\cdots,0) = \vec{a^c}$$

Hence, we complete this proof

$\square$

Proof for Lemma 3.7

Lemma 3.7

Proof.

Write $\vec{x} = (x_1, x_2, \cdots, x_N), \vec{y} = (y_1, y_2, \cdots, y_N) \in A_N$

Proof for Lemma 3.7

Lemma 3.7

Proof.

Write $\vec{x} = (x_1, x_2, \cdots, x_N), \vec{y} = (y_1, y_2, \cdots, y_N) \in A_N$

(a)

$$\begin{aligned} T(c\vec{x} + \vec{y}) &= T(cx_1 + y_1, cx_2 + y_2, \cdots, cx_N + y_N) \\ &= (cx_2 + y_2, \cdots, cx_N + y_N, cx_1 + y_1) \\ &= c(x_2, \cdots, x_N, x_1) + (y_2, \cdots, y_N, y_1) \\ &= cT(\vec{x}) + T(\vec{y}) \end{aligned}$$

Proof for Lemma 3.7

Lemma 3.7

Proof.

Write $\vec{x} = (x_1, x_2, \cdots, x_N), \vec{y} = (y_1, y_2, \cdots, y_N) \in A_N$

(a)

$$\begin{aligned} T(c\vec{x} + \vec{y}) &= T(cx_1 + y_1, cx_2 + y_2, \cdots, cx_N + y_N) \\ &= (cx_2 + y_2, \cdots, cx_N + y_N, cx_1 + y_1) \\ &= c(x_2, \cdots, x_N, x_1) + (y_2, \cdots, y_N, y_1) \\ &= cT(\vec{x}) + T(\vec{y}) \end{aligned}$$

(b)

Given $(a_1, a_2, \cdots, a_N) \in A_N$

$\square$

Proof for Lemma 3.7

(continued...)

(b) Note that

$$\begin{aligned} D \circ T(a_1, a_2, \cdots, a_N) &= D(T(a_1, a_2, \cdots, a_N)) \\ &= D(a_2, \cdots, a_N, a_1) \\ &= (|a_2 - a_3|, \cdots, |a_N - a_1|, \\ &\quad |a_1 - a_2|) \end{aligned}$$

and

$$\begin{aligned} T \circ D(a_1, a_2, \cdots, a_N) &= T(D(a_1, a_2, \cdots, a_N)) \\ &= T(|a_1 - a_2|, \cdots, |a_{N-1} - \\ &\quad a_N|, |a_N - a_1|) \end{aligned}$$

□

Proof for Lemma 3.7

(continued...)

(b)

$$= (|a_2 - a_3|, \cdots, |a_N - a_1|, |a_1 - a_2|)$$

Proof for Lemma 3.7

(continued...)

(b)

$$= (|a_2 - a_3|, \cdots, |a_N - a_1|, |a_1 - a_2|)$$

$$\Longrightarrow \; D \circ T(a_1, a_2, \cdots, a_N) = T \circ D(a_1, a_2, \cdots, a_N)$$

Proof for Lemma 3.7

(continued...)

(b)

$$= (|a_2 - a_3|, \cdots, |a_N - a_1|, |a_1 - a_2|)$$

$$\Longrightarrow \quad D \circ T(a_1, a_2, \cdots, a_N) = T \circ D(a_1, a_2, \cdots, a_N)$$

$$\therefore D \circ T = T \circ D$$

$\square$

Proof for Remark 3.8

Remark 3.8

Proof.

Given $x, y \in \mathbb{Z}_2$

Proof for Remark 3.8

Remark 3.8

Proof.

Given $x, y \in \mathbb{Z}_2$

$\implies 2y = 0$

Proof for Remark 3.8

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Given $x, y \in \mathbb{Z}_2$

$$\implies 2y = 0$$

$$\implies x - y = x - y + 2y = x + y$$

Proof for Remark 3.8

Remark 3.8

Proof.

Given $x, y \in \mathbb{Z}_2$

$$\implies 2y = 0$$

$$\implies x - y = x - y + 2y = x + y$$

$$\implies |x - y| = |x + y|$$

Proof for Remark 3.8

Remark 3.8

Proof.

Given $x, y \in \mathbb{Z}_2$

$\implies 2y = 0$

$\implies x - y = x - y + 2y = x + y$

$\implies |x - y| = |x + y|$

$\implies |x - y| = x + y$, since $x, y \in \mathbb{Z}_2$ $\square$

Proof for Remark 3.9

Remark 3.9

Proof.

"$\Rightarrow$" It is trivial

"$\Leftarrow$" Suppose the condition holds

Proof for Remark 3.9

Remark 3.9

Proof.

"$\Rightarrow$" It is trivial

"$\Leftarrow$" Suppose the condition holds

Given $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$, and $c \in \mathbb{Z}_2$

We must show that $\mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$:

Proof for Remark 3.9

Remark 3.9

Proof.

"$\Rightarrow$" It is trivial

"$\Leftarrow$" Suppose the condition holds
Given $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$, and $c \in \mathbb{Z}_2$
We must show that $\mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$:
If $c = 1$, then there is nothing to prove
Now, we may assume that $c = 0$:

Proof for Remark 3.9

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Given $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$, and $c \in \mathbb{Z}_2$

We must show that $\mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$:

If $c = 1$, then there is nothing to prove

Now, we may assume that $c = 0$:

$\Longrightarrow \mathscr{L}(c\vec{x} + \vec{y}) = \mathscr{L}(\vec{\mathbf{0}} + \vec{y}) = \mathscr{L}(\vec{y})$

Proof for Remark 3.9

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Proof.

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"$\Leftarrow$" Suppose the condition holds
Given $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$, and $c \in \mathbb{Z}_2$
We must show that $\mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$:
If $c = 1$, then there is nothing to prove
Now, we may assume that $c = 0$:
$\Longrightarrow \mathscr{L}(c\vec{x} + \vec{y}) = \mathscr{L}(\vec{\mathbf{0}} + \vec{y}) = \mathscr{L}(\vec{y})$ and
$c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y}) = \vec{\mathbf{0}} + \mathscr{L}(\vec{y}) = \mathscr{L}(\vec{y})$

Proof for Remark 3.9

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"$\Rightarrow$" It is trivial

"$\Leftarrow$" Suppose the condition holds

Given $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$, and $c \in \mathbb{Z}_2$

We must show that $\mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$:

If $c = 1$, then there is nothing to prove

Now, we may assume that $c = 0$:

$\Longrightarrow \mathscr{L}(c\vec{x} + \vec{y}) = \mathscr{L}(\vec{\mathbf{0}} + \vec{y}) = \mathscr{L}(\vec{y})$ and

$c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y}) = \vec{\mathbf{0}} + \mathscr{L}(\vec{y}) = \mathscr{L}(\vec{y})$

$\therefore \mathscr{L}(c\vec{x} + \vec{y}) = c\mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$

$\square$

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:

$i = 0$:

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{T}^0 = \mathscr{I}$ is a linear transformation, holds

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{T}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:
$i = 0$:
$\mathscr{T}^0 = \mathscr{I}$ is a linear transformation, holds
Suppose $i = K$ holds
Then, $i = K + 1$:
By Remark 3.9, it suffices to show that:

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{T}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

By Remark 3.9, it suffices to show that:

$$\mathscr{T}^{K+1}(\vec{x} + \vec{y}) = \mathscr{T}^{K+1}(\vec{x}) + \mathscr{T}^{K+1}(\vec{y}), \forall \vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$$

Proof for Lemma 3.10

Lemma 3.10

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{T}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

By Remark 3.9, it suffices to show that:

$$\mathscr{T}^{K+1}(\vec{x} + \vec{y}) = \mathscr{T}^{K+1}(\vec{x}) + \mathscr{T}^{K+1}(\vec{y}), \forall\, \vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$$

Given $\vec{x} = (x_1, x_2, \cdots, x_N), \vec{y} = (y_1, y_2, \cdots, y_N) \in (\mathbb{Z}_2)^N$ □

Proof for Lemma 3.10

(continued...)

$$\begin{aligned}
\mathscr{T}^{K+1}(\vec{x}+\vec{y}) &= \mathscr{T}^{K}(T(\vec{x}+\vec{y})) \\
&= \mathscr{T}^{K}(\mathscr{T}(x_1+y_1, x_2+y_2, \cdots, x_N+y_N)) \\
&= \mathscr{T}^{K}(x_2+y_2, \cdots, x_N+y_N, x_1+y_1) \\
&= \mathscr{T}^{K}(x_2, \cdots, x_N, x_1) + \mathscr{T}^{K}(y_2, \cdots, y_N), \text{ by} \\
&\quad \text{induction hypothesis} \\
&= \mathscr{T}^{K}(\mathscr{T}(x_1, x_2, \cdots, x_N)) + \mathscr{T}^{K}(\mathscr{T}(y_1, y_2, \cdots, y_N)) \\
&= \mathscr{T}^{K+1}(x_1, x_2, \cdots, x_N) + \mathscr{T}^{K+1}(y_1, y_2, \cdots, y_N) \\
&= \mathscr{T}^{K+1}(\vec{x}) + \mathscr{T}^{K+1}(\vec{y})
\end{aligned}$$

By induction, we complete this proof $\square$

Proof for Lemma 3.11

Lemma 3.11

Proof.

We prove it by induction on i:

$i = 0$:

Proof for Lemma 3.11

Lemma 3.11

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{D}^0 = \mathscr{I}$ is a linear transformation, holds

Proof for Lemma 3.11

Lemma 3.11

Proof.

We prove it by induction on i:

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$\mathscr{D}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

Proof for Lemma 3.11

Lemma 3.11

Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{D}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

By Remark 3.9, it suffices to show that:

$$\mathscr{D}^{K+1}(\vec{x} + \vec{y}) = \mathscr{D}^{K+1}(\vec{x}) + \mathscr{D}^{K+1}(\vec{y}), \forall\, \vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$$

Proof for Lemma 3.11

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Proof.

We prove it by induction on i:

$i = 0$:

$\mathscr{D}^0 = \mathscr{I}$ is a linear transformation, holds

Suppose $i = K$ holds

Then, $i = K + 1$:

By Remark 3.9, it suffices to show that:

$$\mathscr{D}^{K+1}(\vec{x} + \vec{y}) = \mathscr{D}^{K+1}(\vec{x}) + \mathscr{D}^{K+1}(\vec{y}), \forall \vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$$

Given $\vec{x} = (x_1, x_2, \cdots, x_N), \vec{y} = (y_1, y_2, \cdots, y_N) \in (\mathbb{Z}_2)^N$ □

Proof for Lemma 3.11

(continued...)

$$
\begin{aligned}
\mathscr{D}^{K+1}(\vec{x}+\vec{y}) &= \mathscr{D}^{K}(\mathscr{D}(\vec{x}+\vec{y})) \\
&= \mathscr{D}^{K}(\mathscr{D}(x_1+y_1, x_2+y_2, \cdots, x_N+y_N)) \\
&= \mathscr{D}^{K}(\,|(x_1+y_1)-(x_2+y_2)|, \cdots, |(x_{N-1}+y_{N-1}) \\
&\quad -(x_N+y_N)|, |(x_N+y_N)-(x_1+y_1)|\,) \\
&= \mathscr{D}^{K}(\,(x_1+y_1)+(x_2+y_2), \cdots, (x_{N-1}+y_{N-1}) \\
&\quad +(x_N+y_N), (x_N+y_N)+(x_1+y_1)\,), \text{by Remark 3.8} \\
&= \mathscr{D}^{K}((x_1+x_2)+(y_1+y_2), \cdots, (x_{N-1}+x_N) \\
&\quad +(y_{N-1}+y_N), (x_N+x_1)+(y_N+y_1))
\end{aligned}
$$

□

Proof for Lemma 3.11

(continued...)

$$= \mathscr{D}^K(x_1 + x_2, \cdots, x_{N-1} + x_N, x_N + x_1) + \mathscr{D}^K(y_1 + y_2, \cdots, y_{N-1} + y_N, y_N + y_1), \text{ by induction hypothesis}$$
$$= \mathscr{D}^K(\,|x_1 - x_2|, \cdots, |x_{N-1} - x_N|, |x_N - x_1|\,) + \mathscr{D}^K(\,|y_1 - y_2|, \cdots, |y_{N-1} - y_N|, |y_N - y_1|), \text{ by Remark 3.8}$$
$$= \mathscr{D}^K(\mathscr{D}(x_1, x_2, \cdots, x_N)) + \mathscr{D}^K(\mathscr{D}(y_1, y_2, \cdots, y_N))$$
$$= \mathscr{D}^{K+1}(x_1, x_2, \cdots, x_N) + \mathscr{D}^{K+1}\mathscr{D}(y_1, y_2, \cdots, y_N)$$
$$= \mathscr{D}^{K+1}(\vec{x}) + \mathscr{D}^{K+1}(\vec{y})$$

By induction, we complete this proof $\square$

Proof for Lemma 3.12

Lemma 3.12

Proof.

We prove it by induction on i:

Proof for Lemma 3.12

Lemma 3.12

Proof.

We prove it by induction on i:

$i = 0$: It is trivial

Proof for Lemma 3.12

Lemma 3.12

Proof.

We prove it by induction on i:

$i = 0$: It is trivial

Suppose $i = K$ holds

Then, $i = K + 1$:

Proof for Lemma 3.12

Lemma 3.12

Proof.

We prove it by induction on i:
$i = 0$: It is trivial
Suppose $i = K$ holds
Then, $i = K + 1$:

$$\begin{aligned} D^{(r-s)(K+1)}(D^t(\vec{a})) &= D^{r-s}(D^{(r-s)K}(D^t(\vec{a}))) \\ &= D^{r-s}(D^t(\vec{a})), \text{ by induction hypothesis} \\ &= D^{r-s+t}(\vec{a}) \\ &= D^{t-s}(D^r(\vec{a})), \text{ since } s \leq t \\ &= D^{t-s}(D^s(\vec{a})) \\ &= D^t(\vec{a}), \text{ holds} \end{aligned}$$

By induction, we complete this proof □

Proof for Theorem 3.13

Theorem 3.13

Proof.

Note that $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

Proof for Theorem 3.13

Theorem 3.13

Proof.

Note that $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

(a) **Claim:** $D^r(\vec{e}_i) = D^s(\vec{e}_i)$ for each $i = 1, 2, \cdots, N$

Proof.

Given $i \in \mathbb{N}$ with $1 \leq i \leq N$

Proof for Theorem 3.13

Theorem 3.13

Proof.

Note that $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

(a) **Claim:** $D^r(\vec{e}_i) = D^s(\vec{e}_i)$ for each $i = 1, 2, \cdots, N$

Proof.

Given $i \in \mathbb{N}$ with $1 \leq i \leq N$

If $i = 1$, then there is nothing to prove

Now, we may assume that $2 \leq i \leq N$

Proof for Theorem 3.13

Theorem 3.13

Proof.

Note that $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

(a) **Claim:** $D^r(\vec{e}_i) = D^s(\vec{e}_i)$ for each $i = 1, 2, \cdots, N$

Proof.

Given $i \in \mathbb{N}$ with $1 \leq i \leq N$

If $i = 1$, then there is nothing to prove

Now, we may assume that $2 \leq i \leq N$

Note that $\vec{e}_i = T^{N-i+1}(\vec{e}_1)$ □

□

Proof for Theorem 3.13

(continued...)

(a) **Claim:** $D^r(\vec{e}_i) = D^s(\vec{e}_i)$ for each $i = 1, 2, \cdots, N$

(continued...)

$$\begin{aligned} D^r(\vec{e}_i) &= D^r(T^{N-i+1}(\vec{e}_1)) \\ &= T^{N-i+1}(D^r(\vec{e}_1)), \text{ by Lemma 3.7(b)} \\ &= T^{N-i+1}(D^s(\vec{e}_1)) \\ &= D^s(T^{N-i+1}(\vec{e}_1)), \text{ by Lemma 3.7(b)} \\ &= D^s(\vec{e}_i) \end{aligned}$$

$\therefore D^r(\vec{e}_i) = D^s(\vec{e}_i)$ for each $i = 1, 2, \cdots, N$ □

□

Proof for Theorem 3.13

(continued...)

(a) Given $\vec{b} \in (\mathbb{Z}_2)^N$
Finally, we must show that $D^r(\vec{b}) = D^s(\vec{b})$:

Proof for Theorem 3.13

(continued...)

(a) Given $\vec{b} \in (\mathbb{Z}_2)^N$
Finally, we must show that $D^r(\vec{b}) = D^s(\vec{b})$:
$\because \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

Proof for Theorem 3.13

(continued...)

(a) Given $\vec{b} \in (\mathbb{Z}_2)^N$

Finally, we must show that $D^r(\vec{b}) = D^s(\vec{b})$:

$\because \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N\}$ is a basis of $(\mathbb{Z}_2)^N$ over $\mathbb{Z}_2$

$\therefore \exists\, c_1, c_2, \cdots, c_N \in \mathbb{Z}_2$ such that

$$\vec{b} = c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_N\vec{e}_N$$

Then, we have:

$$\begin{aligned}
D^r(\vec{b}) &= \mathscr{D}^r(\vec{b}) \\
&= \mathscr{D}^r(c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_N\vec{e}_N) \\
&= c_1\mathscr{D}^r(\vec{e}_1) + c_2\mathscr{D}^r(\vec{e}_2) + \cdots + c_N\mathscr{D}^r(\vec{e}_N), \\
&\quad \text{by Lemma 3.11} \\
&= c_1 D^r(\vec{e}_1) + c_2 D^r(\vec{e}_2) + \cdots + c_N D^r(\vec{e}_N)
\end{aligned}$$

$\square$

Proof for Theorem 3.13

(continued...)

(a)

$$= c_1 D^r(\vec{e}_1) + c_2 D^r(\vec{e}_2) + \cdots + c_N D^r(\vec{e}_N)$$
$$= c_1 D^s(\vec{e}_1) + c_2 D^s(\vec{e}_2) + \cdots + c_N D^s(\vec{e}_N),$$

by **Claim**

$$= c_1 \mathscr{D}^s(\vec{e}_1) + c_2 \mathscr{D}^s(\vec{e}_2) + \cdots + c_N \mathscr{D}^s(\vec{e}_N)$$
$$= \mathscr{D}^s(c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_N\vec{e}_N),$$ by Lemma 3.11
$$= \mathscr{D}^s(\vec{b})$$
$$= D^s(\vec{b})$$

□

Proof for Theorem 3.13

(continued...)

(b) Given $i \in \mathbb{N}$ with $2 \leq i \leq N$
It suffices to show that the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$:

Proof for Theorem 3.13

(continued...)

(b) Given $i \in \mathbb{N}$ with $2 \leq i \leq N$
It suffices to show that the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$:
By Lemma 2.1, we may assume that the period of $\vec{e}_1 = n - k$

Proof for Theorem 3.13

(continued...)

(b) Given $i \in \mathbb{N}$ with $2 \leq i \leq N$
It suffices to show that the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$:
By Lemma 2.1, we may assume that the period of $\vec{e}_1 = n - k$
$\implies D^0(\vec{e}_1) = \vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), \cdots, D^{n-1}(\vec{e}_1)$ are all distinct

Proof for Theorem 3.13

(continued...)

(b) Given $i \in \mathbb{N}$ with $2 \leq i \leq N$
It suffices to show that the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$:
By Lemma 2.1, we may assume that the period of $\vec{e}_1 = n - k$
$\implies D^0(\vec{e}_1) = \vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), \cdots, D^{n-1}(\vec{e}_1)$ are all distinct and $D^n(\vec{e}_1) = D^k(\vec{e}_1)$

Proof for Theorem 3.13

(continued...)

(b) Given $i \in \mathbb{N}$ with $2 \leq i \leq N$
It suffices to show that the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$:
By Lemma 2.1, we may assume that the period of $\vec{e}_1 = n - k$
$\implies D^0(\vec{e}_1) = \vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), \cdots, D^{n-1}(\vec{e}_1)$ are all distinct and $D^n(\vec{e}_1) = D^k(\vec{e}_1)$
By **(a)**, we know that $D^n(\vec{e}_i) = D^k(\vec{e}_i)$ $(*)$
Claim: $D^0(\vec{e}_i) = \vec{e}_i, D(\vec{e}_i), D^2(\vec{e}_i), \cdots, D^{N-1}(\vec{e}_i)$ are all distinct

$\square$

Proof for Theorem 3.13

(continued...)

(b) **Claim:** $\vec{e}_i, D(\vec{e}_i), D^2(\vec{e}_i), \cdots, D^{N-1}(\vec{e}_i)$ are all distinct

Proof.

If not, suppose $\exists\, a, b \in \mathbb{Z}$ with $0 \leq a < b \leq n-1$ such that
$D^a(\vec{e}_i) = D^b(\vec{e}_i)$
$\implies T^{i-1}(D^a(\vec{e}_i)) = T^{i-1}(D^b(\vec{e}_i))$
$\implies D^a(T^{i-1}(\vec{e}_i)) = D^b(T^{i-1}(\vec{e}_i))$, by Lemma 3.7(b)
$\implies D^a(\vec{e}_1) = D^b(\vec{e}_1)$ which is a contradiction to

$$D^0(\vec{e}_1) = \vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), \cdots, D^{n-1}(\vec{e}_1)$$

are all distinct □

□

Proof for Theorem 3.13

(continued...)

(b) By **Claim** and $(*)$, the period of $\vec{e}_i$ is equal to $n-k$

Proof for Theorem 3.13

(continued...)

(b) By **Claim** and $(*)$, the period of $\vec{e}_i$ is equal to $n-k$
$\implies$ the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$

Proof for Theorem 3.13

(continued...)

(b) By **Claim** and $(*)$, the period of $\vec{e}_i$ is equal to $n-k$
$\implies$ the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$

(c) It is enough to prove that the period of $\vec{a}$ divides the period of $\vec{e}_1$
By Lemma 2.1, we may assume that the periods of $\vec{a}$ and $\vec{e}_1$ are $n-k$ and $n'-k'$, respectively

Proof for Theorem 3.13

(continued...)

(b) By **Claim** and $(*)$, the period of $\vec{e}_i$ is equal to $n-k$
$\Longrightarrow$ the period of $\vec{e}_i$ is equal to the period of $\vec{e}_1$

(c) It is enough to prove that the period of $\vec{a}$ divides the period of $\vec{e}_1$
By Lemma 2.1, we may assume that the periods of $\vec{a}$ and $\vec{e}_1$ are $n-k$ and $n'-k'$, respectively
Write $n'-k' = (n-k)q + r$, where q, r are nonnegative integers with

$$0 \leq r < n-k$$

So, it suffices to show that $r = 0$:

□

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\,\vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$
$\implies D^n(\vec{b}) = D^k(\vec{b})$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\,\vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$
$\implies D^n(\vec{b}) = D^k(\vec{b})$
Moreover, the period of $\vec{e}_1$ is $n' - k'$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

$\implies D^n(\vec{b}) = D^k(\vec{b})$

Moreover, the period of $\vec{e}_1$ is $n' - k'$

$\implies D^{n'}(\vec{e}_1) = D^{k'}(\vec{e}_1)$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$
$\implies D^n(\vec{b}) = D^k(\vec{b})$
Moreover, the period of $\vec{e}_1$ is $n' - k'$
$\implies D^{n'}(\vec{e}_1) = D^{k'}(\vec{e}_1)$
By **(a)**, we obtain $D^{n'}(\vec{b}) = D^{k'}(\vec{b})$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$
$\implies D^n(\vec{b}) = D^k(\vec{b})$
Moreover, the period of $\vec{e}_1$ is $n' - k'$
$\implies D^{n'}(\vec{e}_1) = D^{k'}(\vec{e}_1)$
By **(a)**, we obtain $D^{n'}(\vec{b}) = D^{k'}(\vec{b})$
Take $\vec{\boldsymbol{b}} = D^{k+k'}(\vec{b})$

Proof for Theorem 3.13

(continued...)

(c) By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

$\implies D^n(\vec{b}) = D^k(\vec{b})$

Moreover, the period of $\vec{e}_1$ is $n' - k'$

$\implies D^{n'}(\vec{e}_1) = D^{k'}(\vec{e}_1)$

By **(a)**, we obtain $D^{n'}(\vec{b}) = D^{k'}(\vec{b})$

Take $\vec{\boldsymbol{b}} = D^{k+k'}(\vec{b})$

By Lemma 3.12, we obtain $D^{n'-k'}(\vec{\boldsymbol{b}}) = \vec{\boldsymbol{b}}$

$\implies$

$$\vec{\boldsymbol{b}} = D^{n'-k'}(\vec{\boldsymbol{b}}) = D^{(n-k)q+r}(\vec{\boldsymbol{b}}) = D^r(D^{(n-k)q}(\vec{\boldsymbol{b}}))$$

$$= D^r(\vec{\boldsymbol{b}}), \text{ by Lemma 3.12}$$

$\square$

Proof for Theorem 3.13

(continued...)

(c) $\implies D^{k+k'}(\vec{b}) = D^{k+k'+r}(\vec{b})$

Proof for Theorem 3.13

(continued...)

(c) $\implies D^{k+k'}(\vec{b}) = D^{k+k'+r}(\vec{b})$

Write $k' = (n-k)q_0 + r_0$, where q_0, r_0 are nonnegative integers with $0 \leq r_0 < n-k$

$\implies$

$$\begin{aligned} D^{k+k'}(\vec{b}) &= D^{(k+(n-k)q_0+r_0)}(\vec{b}) \\ &= D^{(n-k)q_0}(D^{k+r_0}(\vec{b})) \\ &= D^{k+r_0}(\vec{b}), \text{ by Lemma 3.12} \end{aligned}$$

and

$\square$

Proof for Theorem 3.13

(continued...)

(c)

$$\begin{aligned} D^{k+k'+r}(\vec{b}) &= D^{k+((n-k)q_0+r_0)+r}(\vec{b}) \\ &= D^{(n-k)q_0}(D^{k+r_0+r}(\vec{b})) \\ &= D^{k+r_0+r}(\vec{b}), \text{ by Lemma 3.12} \end{aligned}$$

Proof for Theorem 3.13

(continued...)

(c)

$$\begin{aligned} D^{k+k'+r}(\vec{b}) &= D^{k+((n-k)q_0+r_0)+r}(\vec{b}) \\ &= D^{(n-k)q_0}(D^{k+r_0+r}(\vec{b})) \\ &= D^{k+r_0+r}(\vec{b}), \text{ by Lemma 3.12} \end{aligned}$$

Then, we have:

$$\begin{aligned} D^{k+r_0+r}(\vec{b}) &= D^{k+k'+r}(\vec{b}) \\ &= D^{k+k'}(\vec{b}) \\ &= D^{k+r_0}(\vec{b}) \end{aligned} \qquad (*')$$

$\square$

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \leq k + r_0 + r < k + r_0 + (n - k) = n + r_0$

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \leq k + r_0 + r < k + r_0 + (n - k) = n + r_0$
$\Longrightarrow k + r_0 \leq k + r_0 + r \leq (n-1) + r_0$, since
$k + r_0 + r, n + r_0$ are integers

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \leq k + r_0 + r < k + r_0 + (n-k) = n + r_0$
$\implies k + r_0 \leq k + r_0 + r \leq (n-1) + r_0$, since
$k + r_0 + r, n + r_0$ are integers
On the other hand, $\vec{b}, D(\vec{b}), \cdots, D^k(\vec{b})$,
$D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ are all distinct

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \le k + r_0 + r < k + r_0 + (n-k) = n + r_0$
$\implies k + r_0 \le k + r_0 + r \le (n-1) + r_0$, since
$k + r_0 + r, n + r_0$ are integers
On the other hand, $\vec{b}, D(\vec{b}), \cdots, D^k(\vec{b})$,
$D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ are all distinct and
$D^n(\vec{b}) = D^k(\vec{b})$, since the period of $\vec{b}$ is $n-k$

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \leq k + r_0 + r < k + r_0 + (n - k) = n + r_0$ $\implies k + r_0 \leq k + r_0 + r \leq (n-1) + r_0$, since $k + r_0 + r, n + r_0$ are integers
On the other hand, $\vec{b}, D(\vec{b}), \cdots, D^k(\vec{b})$, $D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ are all distinct and $D^n(\vec{b}) = D^k(\vec{b})$, since the period of $\vec{b}$ is $n - k$
Then, we have:
$D^{k+r_0}(\vec{b}), D^{k+r_0+1}(\vec{b}), \cdots, D^{k+r_0+r}(\vec{b}), \cdots, D^{(n-1)+r_0}(\vec{b})$ are all distinct

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \le k + r_0 + r < k + r_0 + (n - k) = n + r_0$
$\implies k + r_0 \le k + r_0 + r \le (n-1) + r_0$, since
$k + r_0 + r, n + r_0$ are integers
On the other hand, $\vec{b}, D(\vec{b}), \cdots, D^k(\vec{b})$,
$D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ are all distinct and
$D^n(\vec{b}) = D^k(\vec{b})$, since the period of $\vec{b}$ is $n - k$
Then, we have:
$D^{k+r_0}(\vec{b}), D^{k+r_0+1}(\vec{b}), \cdots, D^{k+r_0+r}(\vec{b}), \cdots, D^{(n-1)+r_0}(\vec{b})$
are all distinct
By $(*')$, we conclude that $k + r_0 = k + r_0 + r$

Proof for Theorem 3.13

(continued...)

(c) Note that $k + r_0 \leq k + r_0 + r < k + r_0 + (n - k) = n + r_0$
$\implies k + r_0 \leq k + r_0 + r \leq (n - 1) + r_0$, since
$k + r_0 + r, n + r_0$ are integers
On the other hand, $\vec{b}, D(\vec{b}), \cdots, D^k(\vec{b})$,
$D^{k+1}(\vec{b}), \cdots, D^{n-1}(\vec{b})$ are all distinct and
$D^n(\vec{b}) = D^k(\vec{b})$, since the period of $\vec{b}$ is $n - k$
Then, we have:
$D^{k+r_0}(\vec{b}), D^{k+r_0+1}(\vec{b}), \cdots, D^{k+r_0+r}(\vec{b}), \cdots, D^{(n-1)+r_0}(\vec{b})$
are all distinct
By $(*')$, we conclude that $k + r_0 = k + r_0 + r$
$\implies r = 0$
Hence, we complete this proof

$\square$

Proof for Lemma 3.14

Lemma 3.14

Proof.

(a) Given $\vec{x} = (x_1, x_2, \cdots, x_N) \in (\mathbb{Z}_2)^N$

Proof for Lemma 3.14

Lemma 3.14

Proof.

(a) Given $\vec{x} = (x_1, x_2, \cdots, x_N) \in (\mathbb{Z}_2)^N$

$$\begin{aligned}
\mathscr{I} + \mathscr{T}(\vec{x}) &= \mathscr{I}(\vec{x}) + \mathscr{T}(\vec{x}) \\
&= \mathscr{I}(x_1, x_2, \cdots, x_N) + \mathscr{T}(x_1, x_2, \cdots, x_N) \\
&= (x_1, x_2, \cdots, x_N) + (x_2, \cdots, x_N, x_1) \\
&= (x_1 + x_2, \cdots, x_{N-1} + x_N, x_N + x_1) \\
&= (\,|x_1 - x_2|, \cdots, |x_{N-1} - x_N|, |x_N - x_1|\,), \\
&\quad \text{by Remark 3.8} \\
&= \mathscr{D}(x_1, x_2, \cdots, x_N) \\
&= \mathscr{D}(\vec{x})
\end{aligned}$$

$$\therefore \mathscr{D} = \mathscr{I} + \mathscr{T}$$

□

Proof for Lemma 3.14

(continued...)

(b) By **(a)**, $\mathscr{D}^{2^r} = (\mathscr{I} + \mathscr{T})^{2^r}$
Claim: $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

Proof.

We prove it by induction on r:

Proof for Lemma 3.14

(continued...)

(b) By **(a)**, $\mathscr{D}^{2^r} = (\mathscr{I} + \mathscr{T})^{2^r}$
Claim: $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

Proof.

We prove it by induction on r:
$r = 0$: It is trivial

Proof for Lemma 3.14

(continued...)

(b) By **(a)**, $\mathscr{D}^{2^r} = (\mathscr{I} + \mathscr{T})^{2^r}$
Claim: $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

Proof.

We prove it by induction on r:
$r = 0$: It is trivial
Suppose $r = K$ holds
Then, $r = K + 1$:

Proof for Lemma 3.14

(continued...)

(b) By **(a)**, $\mathscr{D}^{2^r} = (\mathscr{I} + \mathscr{T})^{2^r}$
Claim: $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

Proof.

We prove it by induction on r:
$r = 0$: It is trivial
Suppose $r = K$ holds
Then, $r = K + 1$:

$$
\begin{aligned}
(\mathscr{I} + \mathscr{T})^{2^{K+1}} &= ((\mathscr{I} + \mathscr{T})^{2^K})^2 \\
&= (\mathscr{I} + \mathscr{T}^{2^K})^2, \text{ by induction hypothesis}
\end{aligned}
$$

□

□

Proof for Lemma 3.14

(continued...)

(b) **Claim:** $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

(continued...)

$$= \mathscr{I}^2 + \mathscr{I}\mathscr{T}^{2^K} + \mathscr{T}^{2^K}\mathscr{I} + (\mathscr{T}^{2^K})^2$$
$$= \mathscr{I} + \mathscr{T}^{2^K} + \mathscr{T}^{2^K} + \mathscr{T}^{2^{K+1}}$$
$$= \mathscr{I} + \mathscr{T}^{2^{K+1}}, \text{ since } \mathscr{T}(\mathbb{Z}_2) \subseteq \mathbb{Z}_2$$

So, $r = K + 1$ holds □

Proof for Lemma 3.14

(continued...)

(b) **Claim:** $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

(continued...)

$$= \mathscr{I}^2 + \mathscr{I}\mathscr{T}^{2^K} + \mathscr{T}^{2^K}\mathscr{I} + (\mathscr{T}^{2^K})^2$$
$$= \mathscr{I} + \mathscr{T}^{2^K} + \mathscr{T}^{2^K} + \mathscr{T}^{2^{K+1}}$$
$$= \mathscr{I} + \mathscr{T}^{2^{K+1}}, \text{ since } \mathscr{T}(\mathbb{Z}_2) \subseteq \mathbb{Z}_2$$

So, $r = K + 1$ holds $\square$

Note that $\mathscr{T}^N = \mathscr{T}$

Proof for Lemma 3.14

(continued...)

(b) **Claim:** $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

(continued...)

$$= \mathscr{I}^2 + \mathscr{I}\mathscr{T}^{2^K} + \mathscr{T}^{2^K}\mathscr{I} + (\mathscr{T}^{2^K})^2$$
$$= \mathscr{I} + \mathscr{T}^{2^K} + \mathscr{T}^{2^K} + \mathscr{T}^{2^{K+1}}$$
$$= \mathscr{I} + \mathscr{T}^{2^{K+1}}, \text{ since } \mathscr{T}(\mathbb{Z}_2) \subseteq \mathbb{Z}_2$$

So, $r = K + 1$ holds $\square$

Note that $\mathscr{T}^N = \mathscr{T}$

By assumption, we know that $\mathscr{T}^{2^r} = \mathscr{T}^s$

Proof for Lemma 3.14

(continued...)

(b) **Claim:** $(\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{I}^{2^r}$

(continued...)

$$= \mathscr{I}^2 + \mathscr{I}\mathscr{T}^{2^K} + \mathscr{T}^{2^K}\mathscr{I} + (\mathscr{T}^{2^K})^2$$
$$= \mathscr{I} + \mathscr{T}^{2^K} + \mathscr{T}^{2^K} + \mathscr{T}^{2^{K+1}}$$
$$= \mathscr{I} + \mathscr{T}^{2^{K+1}}, \text{ since } \mathscr{T}(\mathbb{Z}_2) \subseteq \mathbb{Z}_2$$

So, $r = K + 1$ holds $\square$

Note that $\mathscr{T}^N = \mathscr{T}$

By assumption, we know that $\mathscr{T}^{2^r} = \mathscr{T}^s$

$\therefore \mathscr{D}^{2^r} = (\mathscr{I} + \mathscr{T})^{2^r} = \mathscr{I} + \mathscr{T}^{2^r} = \mathscr{I} + \mathscr{T}^s$

Hence, we complete this proof $\square$

Proof for Theorem 3.15

Theorem 3.15

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$

Proof for Theorem 3.15

Theorem 3.15

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$
Note that $2^r \equiv 0 \pmod{N}$

Proof for Theorem 3.15

Theorem 3.15

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n - k$
Note that $2^r \equiv 0 \pmod{N}$
By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$

Proof for Theorem 3.15

Theorem 3.15

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$
Note that $2^r \equiv 0 \pmod{N}$
By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$

Proof for Theorem 3.15

Theorem 3.15

Proof.

By Lemma 2.1, we may assume that the period of $\vec{a}$ is $n-k$
Note that $2^r \equiv 0 \pmod{N}$
By Theorem 3.2, $\exists\, \vec{b} \in (\mathbb{Z}_2)^N$ with the period of $\vec{b}$ which is equal to the period of $\vec{a}$ such that the cycle of $\vec{a}$ is similar to the cycle of $\vec{b}$
$\implies \exists\, m \in \mathbb{N}$ such that $D^r(\vec{a}) = mD^s(\vec{b})$, where r, s are nonnegative integers with $k \leq s \leq n-1$
$\because$

$$\begin{aligned} D^N(\vec{b}) &= \mathscr{D}^N(\vec{b}), \text{ since } \vec{b} \in (\mathbb{Z}_2)^N \\ &= \mathscr{I} + \mathscr{T}^0(\vec{b}), \text{ by Lemma 3.14} \end{aligned}$$

$\square$

Proof for Theorem 3.15

(continued...)

$$
\begin{aligned}
&= \mathscr{I} + \mathscr{I}(\vec{b}) \\
&= \mathscr{I}(\vec{b}) + \mathscr{I}(\vec{b}) \\
&= \vec{\mathbf{0}}, \text{ since } \mathscr{I}(\vec{b}) \in (\mathbb{Z}_2)^6
\end{aligned}
$$

Proof for Theorem 3.15

(continued...)

$$= \mathscr{I} + \mathscr{I}(\vec{b})$$
$$= \mathscr{I}(\vec{b}) + \mathscr{I}(\vec{b})$$
$$= \vec{\mathbf{0}}, \text{ since } \mathscr{I}(\vec{b}) \in (\mathbb{Z}_2)^6$$

$\therefore$

$$\begin{aligned} D^{r+N}(\vec{a}) &= D^N(D^r(\vec{a})) \\ &= D^N(mD^s(\vec{b})) \\ &= mD^{N+s}(\vec{b}) \\ &= mD^s(D^N(\vec{b})) \\ &= mD^s(\vec{\mathbf{0}}) \end{aligned}$$

$\square$

Proof for Theorem 3.15

(continued...)

$$
\begin{aligned}
&= m\vec{\mathbf{0}} \\
&= \vec{\mathbf{0}}
\end{aligned}
$$

Proof for Theorem 3.15

(continued...)

$$= m\vec{\mathbf{0}}$$
$$= \vec{\mathbf{0}}$$

On the other hand, we know that
$D(\vec{\mathbf{0}}) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = \vec{\mathbf{0}} = D^0(\vec{\mathbf{0}})$

Proof for Theorem 3.15

(continued...)

$$= m\vec{\mathbf{0}}$$
$$= \vec{\mathbf{0}}$$

On the other hand, we know that
$D(\vec{\mathbf{0}}) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = \vec{\mathbf{0}} = D^0(\vec{\mathbf{0}})$
$\Longrightarrow$ the period of $\vec{\mathbf{0}}$ is $1 - 0 = 1$ and the 1-cycle of $\vec{\mathbf{0}}$ is $\vec{\mathbf{0}}$

Proof for Theorem 3.15

(continued...)

$$= m\vec{\mathbf{0}}$$
$$= \vec{\mathbf{0}}$$

On the other hand, we know that
$D(\vec{\mathbf{0}}) = D(0, 0, \cdots, 0) = (0, 0, \cdots, 0) = \vec{\mathbf{0}} = D^0(\vec{\mathbf{0}})$
$\implies$ the period of $\vec{\mathbf{0}}$ is $1 - 0 = 1$ and the 1-cycle of $\vec{\mathbf{0}}$ is $\vec{\mathbf{0}}$
$\implies$ the cycle of $\vec{a}$ is similar to the 1-cycle of $\vec{\mathbf{0}}$ $\square$

Proof for Theorem 4.1

Theorem 4.1

Proof.

Given $\vec{a} \in A_6$

Proof for Theorem 4.1

Theorem 4.1

Proof.

Given $\vec{a} \in A_6$

Let $\vec{e}_1 = (1, 0, 0, 0, 0, 0)$

Proof for Theorem 4.1

Theorem 4.1

Proof.

Given $\vec{a} \in A_6$

Let $\vec{e}_1 = (1,0,0,0,0,0)$

$\Longrightarrow$

$$\begin{aligned} D(\vec{e}_1) &= (1,0,0,0,0,1) \\ D^2(\vec{e}_1) &= (1,0,0,0,1,0) \\ D^3(\vec{e}_1) &= (1,0,0,1,1,1) \\ D^4(\vec{e}_1) &= (1,0,1,0,0,0) \\ D^5(\vec{e}_1) &= (1,1,1,0,0,1) \\ D^6(\vec{e}_1) &= (0,0,1,0,1,0) \end{aligned}$$

$\square$

Proof for Theorem 4.1

(continued...)

$$
\begin{aligned}
D^7(\vec{e}_1) &= (0, 1, 1, 1, 1, 0) \\
D^8(\vec{e}_1) &= (1, 0, 0, 0, 1, 0) \\
&= D^2(\vec{e}_1)
\end{aligned}
$$

Proof for Theorem 4.1

(continued...)

$$\begin{aligned} D^7(\vec{e}_1) &= (0,1,1,1,1,0) \\ D^8(\vec{e}_1) &= (1,0,0,0,1,0) \\ &= D^2(\vec{e}_1) \end{aligned}$$

$\Longrightarrow$ the period of $\vec{e}_1$ is $(8-2) = 6$

Proof for Theorem 4.1

(continued...)

$$\begin{aligned} D^7(\vec{e}_1) &= (0,1,1,1,1,0) \\ D^8(\vec{e}_1) &= (1,0,0,0,1,0) \\ &= D^2(\vec{e}_1) \end{aligned}$$

$\Longrightarrow$ the period of $\vec{e}_1$ is $(8-2)=6$

By Theorem 3.13(c), the period of $\vec{a}$ divides 6 and the maximal period of 6-tuples in A_6 is equal to 6 $\square$

Proof for Lemma 4.2

Lemma 4.2

Proof.

We prove it by enumerating as shown in the following diagrams:

Proof for Lemma 4.2

Lemma 4.2

Proof.

We prove it by enumerating as shown in the following diagrams:

$$\mathbf{(0,0,0,0,0,0)}$$

$\rightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

Lemma 4.2

Proof.

We prove it by enumerating as shown in the following diagrams:

$$
\begin{array}{ccc}
(0,1,0,1,0,1) & & (1,0,1,0,1,0) \\
\searrow & & \swarrow \\
& (1,1,1,1,1,1) & \\
& \downarrow & \\
& \mathbf{(0,0,0,0,0,0)} & \\
& \circlearrowleft &
\end{array}
$$

$\rightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

(continued...)

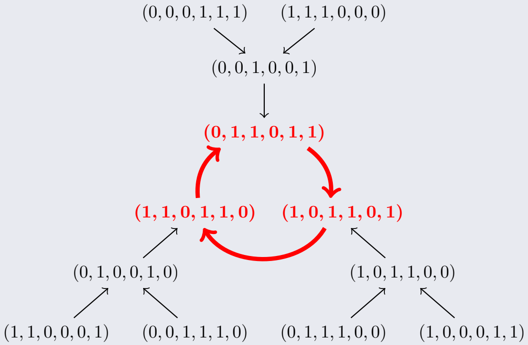
$$\begin{tikzcd} & (0,1,1,0,1,1) \arrow[dr, bend left] & \\ (1,1,0,1,1,0) \arrow[ur, bend left] & & (1,0,1,1,0,1) \arrow[ll, bend left] \end{tikzcd}$$

$\rightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

(continued...)



$\longrightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

(continued...)

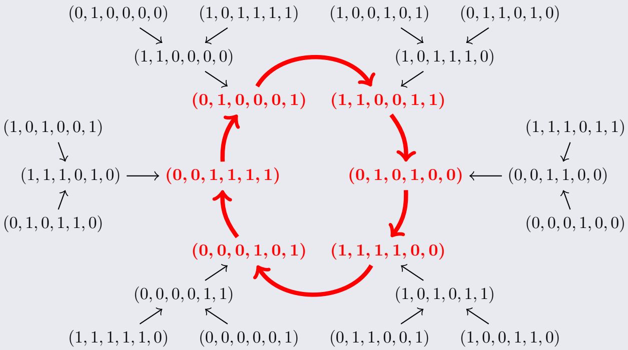
$$
\begin{tikzcd}
(0,1,0,0,0,1) \arrow[r, bend left] & (1,1,0,0,1,1) \arrow[d, bend left] \\
(0,0,1,1,1,1) \arrow[u, bend left] & (0,1,0,1,0,0) \arrow[d, bend left] \\
(0,0,0,1,0,1) \arrow[u, bend left] & (1,1,1,1,0,0) \arrow[l, bend left]
\end{tikzcd}
$$

$\rightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

(continued...)



$\longrightarrow$: a Ducci process

Proof for Lemma 4.2

(continued...)

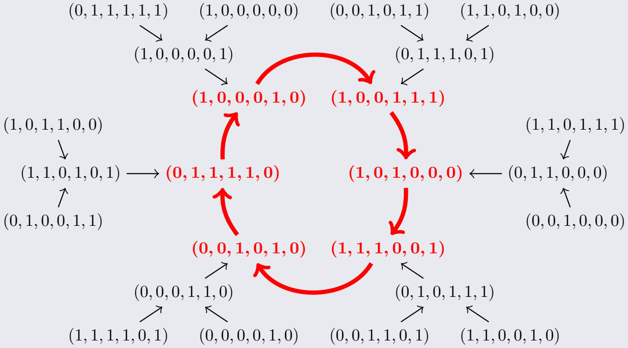
$$
\begin{tikzcd}
 & (1,0,0,0,1,0) \arrow[r, bend left] & (1,0,0,1,1,1) \arrow[dr] & \\
(0,1,1,1,1,0) \arrow[ur] & & & (1,0,1,0,0,0) \arrow[dl] \\
 & (0,0,1,0,1,0) \arrow[ul] & (1,1,1,0,0,1) \arrow[l, bend left] &
\end{tikzcd}
$$

$\rightarrow$: a Ducci process

$\square$

Proof for Lemma 4.2

(continued...)



$\longrightarrow$: a Ducci process

$\square$

Proof for Theorem 4.3

Theorem 4.3

Proof.

It follows from Theorem 3.2 and Lemma 4.2. □

Proof for Lemma 4.4

Lemma 4.4

Proof.

Write $e = (1)(2)(3)(4)(5)(6)$ which is the identity element of $\mathcal{D}_6$

Proof for Lemma 4.4

Lemma 4.4

Proof.

Write $e = (1)(2)(3)(4)(5)(6)$ which is the identity element of $\mathcal{D}_6$

Claim 1: $e * \vec{a} = \vec{a},\ \forall\, \vec{a} \in A_6$

Proof for Lemma 4.4

Lemma 4.4

Proof.

Write $e = (1)(2)(3)(4)(5)(6)$ which is the identity element of $\mathcal{D}_6$

Claim 1: $e * \vec{a} = \vec{a}$, $\forall\, \vec{a} \in A_6$

Proof.

Given $\vec{a} = (a_1, a_2, \cdots, a_6) \in A_6$

$$\begin{aligned} e * \vec{a} &= \left(a_{e(1)}, a_{e(2)}, \cdots, a_{e(6)}\right) \\ &= (a_1, a_2, \cdots, a_6) \\ &= \vec{a} \end{aligned}$$

□

□

Proof for Lemma 4.4

(continued...)

Claim 2: $(\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$, $\forall \pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} \in A_6$

Proof for Lemma 4.4

(continued...)

Claim 2: $(\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$, $\forall\, \pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} \in A_6$

Proof.

Given $\pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} = (a_1, a_2, \cdots, a_6) \in A_6$

Proof for Lemma 4.4

(continued...)

Claim 2: $(\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$, $\forall \pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} \in A_6$

Proof.

Given $\pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} = (a_1, a_2, \cdots, a_6) \in A_6$
Note that

$$(\pi_1 \circ \pi_2) * \vec{a} = \left(a_{\pi_1 \circ \pi_2(1)}, a_{\pi_1 \circ \pi_2(2)}, \cdots, a_{\pi_1 \circ \pi_2(6)}\right)$$

Proof for Lemma 4.4

(continued...)

Claim 2: $(\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$, $\forall \pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} \in A_6$

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Given $\pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} = (a_1, a_2, \cdots, a_6) \in A_6$
Note that

$$(\pi_1 \circ \pi_2) * \vec{a} = \left(a_{\pi_1 \circ \pi_2(1)}, a_{\pi_1 \circ \pi_2(2)}, \cdots, a_{\pi_1 \circ \pi_2(6)}\right)$$

and

$$\begin{aligned} \pi_1 * (\pi_2 * \vec{a}) &= \pi_1 * \left(a_{\pi_2(1)}, a_{\pi_2(2)}, \cdots, a_{\pi_2(6)}\right) \\ &= \left(a_{\pi_1(\pi_2(1))}, a_{\pi_1(\pi_2(2))}, \cdots, a_{\pi_1(\pi_2(6))}\right) \\ &= \left(a_{\pi_1 \circ \pi_2(1)}, a_{\pi_1 \circ \pi_2(2)}, \cdots, a_{\pi_1 \circ \pi_2(6)}\right) \end{aligned}$$

Proof for Lemma 4.4

(continued...)

Claim 2: $(\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$, $\forall \pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} \in A_6$

Proof.

Given $\pi_1, \pi_2 \in \mathcal{D}_6$ and $\vec{a} = (a_1, a_2, \cdots, a_6) \in A_6$
Note that

$$(\pi_1 \circ \pi_2) * \vec{a} = \left(a_{\pi_1 \circ \pi_2(1)}, a_{\pi_1 \circ \pi_2(2)}, \cdots, a_{\pi_1 \circ \pi_2(6)}\right)$$

and

$$\begin{aligned}\pi_1 * (\pi_2 * \vec{a}) &= \pi_1 * \left(a_{\pi_2(1)}, a_{\pi_2(2)}, \cdots, a_{\pi_2(6)}\right) \\ &= \left(a_{\pi_1(\pi_2(1))}, a_{\pi_1(\pi_2(2))}, \cdots, a_{\pi_1(\pi_2(6))}\right) \\ &= \left(a_{\pi_1 \circ \pi_2(1)}, a_{\pi_1 \circ \pi_2(2)}, \cdots, a_{\pi_1 \circ \pi_2(6)}\right)\end{aligned}$$

$\therefore (\pi_1 \circ \pi_2) * \vec{a} = \pi_1 * (\pi_2 * \vec{a})$ □

By **Claim 1** and **Claim 2**, we complete this proof □

Proof for Remark 4.7

Remark 4.7

Proof.

By assumption, we know that $a_1, a_2, \cdots, a_6 \in \{0, 1\}$

Proof for Remark 4.7

Remark 4.7

Proof.

By assumption, we know that $a_1, a_2, \cdots, a_6 \in \{0, 1\}$
$\Longrightarrow 1 - a_1, 1 - a_2, \cdots, 1 - a_6 \in \{0, 1\}$

Proof for Remark 4.7

Remark 4.7

Proof.

By assumption, we know that $a_1, a_2, \cdots, a_6 \in \{0, 1\}$

$$\implies 1 - a_1, 1 - a_2, \cdots, 1 - a_6 \in \{0, 1\}$$

$$\implies \vec{a^c} = (1 - a_1, 1 - a_2, \cdots, 1 - a_6) \in (\mathbb{Z}_2)^6 \qquad \square$$

Proof for Lemma 4.8

Lemma 4.8

Proof.

Write $\vec{a^c} = (b_1, b_2, \cdots, b_6)$

$\implies b_i = 1 - a_i, \forall\, i = 1, 2, \cdots, 6$

Given $\pi \in \mathcal{D}_6$

Proof for Lemma 4.8

Lemma 4.8

Proof.

Write $\vec{a^c} = (b_1, b_2, \cdots, b_6)$
$\implies b_i = 1 - a_i, \forall\, i = 1, 2, \cdots, 6$
Given $\pi \in \mathcal{D}_6$
Note that

$$\begin{aligned} \pi * \vec{a^c} &= (b_{\pi(1)}, b_{\pi(2)}, \cdots, b_{\pi(6)}) \\ &= (1 - a_{\pi(1)}, 1 - a_{\pi(2)}, \cdots, 1 - a_{\pi(6)}) \end{aligned}$$

Proof for Lemma 4.8

Lemma 4.8

Proof.

Write $\vec{a^c} = (b_1, b_2, \cdots, b_6)$
$\implies b_i = 1 - a_i, \forall\, i = 1, 2, \cdots, 6$
Given $\pi \in \mathcal{D}_6$
Note that

$$\begin{aligned}\pi * \vec{a^c} &= \left(b_{\pi(1)}, b_{\pi(2)}, \cdots, b_{\pi(6)}\right)\\ &= \left(1 - a_{\pi(1)}, 1 - a_{\pi(2)}, \cdots, 1 - a_{\pi(6)}\right)\end{aligned}$$

and

$$\begin{aligned}(\pi * \vec{a})^c &= \left(a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(6)}\right)^c\\ &= \left(1 - a_{\pi(1)}, 1 - a_{\pi(2)}, \cdots, 1 - a_{\pi(6)}\right)\end{aligned}$$

Proof for Lemma 4.8

Lemma 4.8

Proof.

Write $\vec{a^c} = (b_1, b_2, \cdots, b_6)$
$\implies b_i = 1 - a_i, \forall\, i = 1, 2, \cdots, 6$
Given $\pi \in \mathcal{D}_6$
Note that

$$\begin{aligned}\pi * \vec{a^c} &= (b_{\pi(1)}, b_{\pi(2)}, \cdots, b_{\pi(6)}) \\ &= (1 - a_{\pi(1)}, 1 - a_{\pi(2)}, \cdots, 1 - a_{\pi(6)})\end{aligned}$$

and

$$\begin{aligned}(\pi * \vec{a})^c &= (a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(6)})^c \\ &= (1 - a_{\pi(1)}, 1 - a_{\pi(2)}, \cdots, 1 - a_{\pi(6)})\end{aligned}$$

$\therefore \pi * \vec{a^c} = (\pi * \vec{a})^c$ $\square$

Proof for Lemma 4.9

Lemma 4.9

Proof.

For each $\pi \in \mathcal{D}_6$, let $Z_\pi = \{z \in (\mathbb{Z}_2)^6 \mid \pi * z = z\}$

Proof for Lemma 4.9

Lemma 4.9

Proof.

For each $\pi \in \mathcal{D}_6$, let $Z_\pi = \{z \in (\mathbb{Z}_2)^6 \mid \pi * z = z\}$
By the Burnside's Lemma, we obtain:

$$\begin{aligned}
|(\mathbb{Z}_2)^6/\equiv| &= \frac{1}{|\mathcal{D}_6|}\sum_{\pi\in\mathcal{D}_6}|Z_\pi| \\
&= \frac{1}{12}(2^6+2+2^2+2^3+2^2+2+2^3+2^4+2^3+2^4+2^3+2^4) \\
&= \frac{1}{12}(64+2+4+8+4+2+8+16+8+16+8+16) \\
&= \frac{1}{12}\cdot 156 \\
&= 13
\end{aligned}$$

$\square$

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)]$, $[(0,0,1,0,0,1)]$, $[(1,0,0,0,0,1)]$, $[(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)]$, $[(0,0,1,0,1,1)]$, $[(0,1,0,1,0,1)]$:

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)], [(0,0,1,0,0,1)], [(1,0,0,0,0,1)], [(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)], [(0,0,1,0,1,1)], [(0,1,0,1,0,1)]$:
$[(0,0,0,0,0,0)] = \{(0,0,0,0,0,0)\}$

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)], [(0,0,1,0,0,1)], [(1,0,0,0,0,1)], [(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)], [(0,0,1,0,1,1)], [(0,1,0,1,0,1)]$:
$[(0,0,0,0,0,0)] = \{(0,0,0,0,0,0)\}$
$[(1,0,0,0,0,0)] = \{(1,0,0,0,0,0), (0,1,0,0,0,0)$, $(0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)], [(0,0,1,0,0,1)], [(1,0,0,0,0,1)], [(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)], [(0,0,1,0,1,1)], [(0,1,0,1,0,1)]$:

$[(0,0,0,0,0,0)] = \{(0,0,0,0,0,0)\}$

$[(1,0,0,0,0,0)] = \{(1,0,0,0,0,0),(0,1,0,0,0,0),$
$(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$[(0,0,1,0,0,1)] = \{(0,0,1,0,0,1),(1,0,0,1,0,0),(0,1,0,0,1,0)\}$

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)], [(0,0,1,0,0,1)], [(1,0,0,0,0,1)], [(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)], [(0,0,1,0,1,1)], [(0,1,0,1,0,1)]$:

$[(0,0,0,0,0,0)] = \{(0,0,0,0,0,0)\}$

$[(1,0,0,0,0,0)] = \{(1,0,0,0,0,0),(0,1,0,0,0,0),$
$(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$[(0,0,1,0,0,1)] = \{(0,0,1,0,0,1),(1,0,0,1,0,0),(0,1,0,0,1,0)\}$

$[(1,0,0,0,0,1)] = \{(1,0,0,0,0,1),(1,1,0,0,0,0),$
$(0,1,1,0,0,0),(0,0,1,1,0,0),(0,0,0,1,1,0),(0,0,0,0,1,1)\}$

Proof for Lemma 4.9

(continued...)

Finally, we enumerate 13 equivalence classes of $(\mathbb{Z}_2)^6$:
By Lemma 4.8, it is reduced to write out $[(0,0,0,0,0,0)]$, $[(1,0,0,0,0,0)], [(0,0,1,0,0,1)], [(1,0,0,0,0,1)], [(1,0,0,0,1,0)]$, $[(0,0,0,1,1,1)], [(0,0,1,0,1,1)], [(0,1,0,1,0,1)]$:

$[(0,0,0,0,0,0)] = \{(0,0,0,0,0,0)\}$

$[(1,0,0,0,0,0)] = \{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$[(0,0,1,0,0,1)] = \{(0,0,1,0,0,1),(1,0,0,1,0,0),(0,1,0,0,1,0)\}$

$[(1,0,0,0,0,1)] = \{(1,0,0,0,0,1),(1,1,0,0,0,0),(0,1,1,0,0,0),(0,0,1,1,0,0),(0,0,0,1,1,0),(0,0,0,0,1,1)\}$

$[(1,0,0,0,1,0)] = \{(1,0,0,0,1,0),(0,1,0,0,0,1),(1,0,1,0,0,0),(0,1,0,1,0,0),(0,0,1,0,1,0),(0,0,0,1,0,1)\}$ $\square$

Proof for Lemma 4.9

(continued..)

$$[(0,0,0,1,1,1)] = \{(0,0,0,1,1,1),(1,0,0,0,1,1),\\(1,1,0,0,0,1),(1,1,1,0,0,0),(0,1,1,1,0,0),(0,0,1,1,1,0)\}$$

Proof for Lemma 4.9

(continued..)

$[(0,0,0,1,1,1)] = \{(0,0,0,1,1,1),(1,0,0,0,1,1),$
$(1,1,0,0,0,1),(1,1,1,0,0,0),(0,1,1,1,0,0),(0,0,1,1,1,0)\}$
$[(0,0,1,0,1,1)] = \{(0,0,1,0,1,1),(1,0,0,1,0,1),$
$(1,1,0,0,1,0),(0,1,1,0,0,1),(1,0,1,1,0,0),(0,1,0,1,1,0),$
$(1,1,0,1,0,0),(0,1,1,0,1,0),(0,0,1,1,0,1),(1,0,0,1,1,0),$
$(0,1,0,0,1,1),(1,0,1,0,0,1)\}$

Proof for Lemma 4.9

(continued..)

$[(0,0,0,1,1,1)] = \{(0,0,0,1,1,1),(1,0,0,0,1,1),$
$(1,1,0,0,0,1),(1,1,1,0,0,0),(0,1,1,1,0,0),(0,0,1,1,1,0)\}$
$[(0,0,1,0,1,1)] = \{(0,0,1,0,1,1),(1,0,0,1,0,1),$
$(1,1,0,0,1,0),(0,1,1,0,0,1),(1,0,1,1,0,0),(0,1,0,1,1,0),$
$(1,1,0,1,0,0),(0,1,1,0,1,0),(0,0,1,1,0,1),(1,0,0,1,1,0),$
$(0,1,0,0,1,1),(1,0,1,0,0,1)\}$
$[(0,1,0,1,0,1)] = \{(0,1,0,1,0,1),(1,0,1,0,1,0)\}$ $\square$

Proof for Theorem 4.10

Theorem 4.10

Proof.

$$\mathbf{(0,0,0,0,0,0)} \circlearrowleft$$

$\rightarrow$: a Ducci process

$\square$

Proof for Theorem 4.10

Theorem 4.10

Proof.

$$(0,1,0,1,0,1)$$
$$\downarrow$$
$$(1,1,1,1,1,1)$$
$$\downarrow$$
$$\mathbf{(0,0,0,0,0,0)}$$
$$\circlearrowleft$$

$\longrightarrow$: a Ducci process

$\square$

Proof for Theorem 4.10

Theorem 4.10

continued...

$$\mathbf{(0, 1, 1, 0, 1, 1)}$$

$$\circlearrowleft$$

$\rightarrow$: a Ducci process

$\square$

Proof for Theorem 4.10

Theorem 4.10

continued...

$$(0,0,0,1,1,1)$$
$$\downarrow$$
$$(0,0,1,0,0,1)$$
$$\downarrow$$
$$\mathbf{(0,1,1,0,1,1)}$$
$$\circlearrowleft$$

$\longrightarrow$: a Ducci process

$\square$

Proof for Theorem 4.10

Theorem 4.10

continued...

$$\begin{tikzcd}[column sep=large] \mathbf{(1,0,0,0,1,0)} \arrow[r, bend left] & \mathbf{(1,0,0,1,1,1)} \arrow[l, bend left] \end{tikzcd}$$

$\rightarrow$: a Ducci process

$\square$

Proof for Theorem 4.10

Theorem 4.10

continued...

$$
\begin{tikzcd}
(0,1,1,1,1,1) \arrow[dr] & & (1,0,0,0,0,0) \arrow[dl] & & & & (0,1,1,1,0,1) \arrow[dl] \\
& (0,1,1,1,0,1) \arrow[dr] & & & & (0,1,1,1,0,1) \arrow[dl] & \\
& & \color{red}{\mathbf{(1,0,0,0,1,0)}} \arrow[rr, bend left, red] & & \color{red}{\mathbf{(1,0,0,1,1,1)}} \arrow[ll, bend left, red] & &
\end{tikzcd}
$$

$\rightarrow$: a Ducci process

$\square$

Proof for Corollary 4.11

Corollary 4.11

Proof.

It follows from Theorem 3.2 and Theorem 4.10. $\square$

Proof for Theorem 4.12

Theorem 4.12

Proof.

"$\Rightarrow$" Suppose that r is the period of $\vec{a}$ for some $\vec{a} \in A_6$

Proof for Theorem 4.12

Theorem 4.12

Proof.

"$\Rightarrow$" Suppose that r is the period of $\vec{a}$ for some $\vec{a} \in A_6$
By Theorem 3.13(c), the period of $\vec{a}$ divides the period of $\vec{e}_1$
Therefore, r divides the period of $\vec{e}_1$

Proof for Theorem 4.12

Theorem 4.12

Proof.

"$\Rightarrow$" Suppose that r is the period of $\vec{a}$ for some $\vec{a} \in A_6$
By Theorem 3.13(c), the period of $\vec{a}$ divides the period of $\vec{e}_1$
Therefore, r divides the period of $\vec{e}_1$

"$\Leftarrow$" Suppose the condition holds

Proof for Theorem 4.12

Theorem 4.12

Proof.

"$\Rightarrow$" Suppose that r is the period of $\vec{a}$ for some $\vec{a} \in A_6$
By Theorem 3.13(c), the period of $\vec{a}$ divides the period of $\vec{e}_1$
Therefore, r divides the period of $\vec{e}_1$

"$\Leftarrow$" Suppose the condition holds
Note that

$$
\begin{aligned}
D(\vec{e}_1) &= (1,0,0,0,0,1)\\
D^2(\vec{e}_1) &= (1,0,0,0,1,0)\\
D^3(\vec{e}_1) &= (1,0,0,1,1,1)\\
D^4(\vec{e}_1) &= (1,0,1,0,0,0)\\
&= (1,0,0,0,1,0), \text{ by the proof in Lemma 4.9}\\
&= D^2(\vec{e}_1)
\end{aligned}
$$

□

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that $\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that $\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct
$\therefore$ the period of $\vec{e}_1$ is $(4-2)=2$

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that $\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct
$\therefore$ the period of $\vec{e}_1$ is $(4-2) = 2$
By assumption, we know that $r \mid 2$
So, we have the following two cases:

Case 1: $r = 1$

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that $\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct

$\therefore$ the period of $\vec{e}_1$ is $(4-2)=2$

By assumption, we know that $r \mid 2$

So, we have the following two cases:

Case 1: $r=1$ Choose

$\vec{a}=(0,0,0,0,0,0) \in (\mathbb{Z}_2)^6 \subset A_6$

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that
$\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct
$\therefore$ the period of $\vec{e}_1$ is $(4-2)=2$
By assumption, we know that $r \mid 2$
So, we have the following two cases:

Case 1: $r=1$ Choose
$\vec{a}=(0,0,0,0,0,0) \in (\mathbb{Z}_2)^6 \subset A_6$
By Theorem 4.10, the cycle of $\vec{a}$ is
$(0,0,0,0,0,0)$ and the period of $\vec{a}$ is 1

Proof for Theorem 4.12

(continued...)

"$\Leftarrow$" By Remark 4.5(a) and Lemma 4.9, we know that
$\vec{e}_1, D(\vec{e}_1), D^2(\vec{e}_1), D^3(\vec{e}_1)$ are all distinct
$\therefore$ the period of $\vec{e}_1$ is $(4-2)=2$
By assumption, we know that $r \mid 2$
So, we have the following two cases:

Case 1: $r=1$ Choose
$\vec{a}=(0,0,0,0,0,0) \in (\mathbb{Z}_2)^6 \subset A_6$
By Theorem 4.10, the cycle of $\vec{a}$ is
$(0,0,0,0,0,0)$ and the period of $\vec{a}$ is 1
$\Longrightarrow$ the period of $\vec{a}$ is r

$\square$

Proof for Theorem 4.2

(continued...)

"$\Leftarrow$" Case 2: $r = 2$

Take $\vec{a} = (0,0,1,0,1,0) \in (\mathbb{Z}_2)^6 \subset A_6$

By Theorem 4.10, the cycle of $\vec{a}$ is $(0,0,1,0,1,0), (0,1,1,1,1,0)$ and the period of $\vec{a}$ is 2

Proof for Theorem 4.2

(continued...)

"$\Leftarrow$" Case 2: $r = 2$

Take $\vec{a} = (0, 0, 1, 0, 1, 0) \in (\mathbb{Z}_2)^6 \subset A_6$

By Theorem 4.10, the cycle of $\vec{a}$ is $(0, 0, 1, 0, 1, 0), (0, 1, 1, 1, 1, 0)$ and the period of $\vec{a}$ is 2

$\Longrightarrow$ the period of $\vec{a}$ is r

By **Case 1 and 2**, we complete this proof

$\square$

Prepare for Lemma 2.6

Remark (2.4)

If $0 \le x, y \le M$, then $|x - y| \le M$.

Proof

Prepare for Theorem 2.12

Remark (2.7)

If $0 \leq x, y \leq M$ with $|x - y| = M$, then $x, y \in \{0, M\}$ and at least one of them is M.

Proof

Prepare for Theorem 2.12

Lemma (2.8)

Let $\vec{a} = (a_1, a_2, \cdots, a_N)$, $\vec{b} = (b_1, b_2, \cdots, b_N) \in A_N$ *such that* $D(\vec{b}) = \vec{a}$ *and* $\max \vec{a} = \max \vec{b} = M$.

Prepare for Theorem 2.12

Lemma (2.8)

Let $\vec{a} = (a_1, a_2, \cdots, a_N)$, $\vec{b} = (b_1, b_2, \cdots, b_N) \in A_N$ *such that* $D(\vec{b}) = \vec{a}$ *and* $\max \vec{a} = \max \vec{b} = M$.
If $a_i \in \{0, M\}$, $\forall\, i = 1, 2, \cdots, t$ *and at least one of them is* M, *then* $b_i \in \{0, M\}$, $\forall\, i = 1, 2, \cdots, t, t+1$ *and at least one of them is* M.

Proof

A remark about Lemma 2.8

Remark (2.9)

In Lemma 2.8, we know that $1 \leq t \leq N-1$, since A_N is a collection of N-tuples of nonnegative integers.

Prepare for Theorem 2.12

Lemma (2.10)

Let $\vec{a} \in A_N$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$.

Prepare for Theorem 2.12

Lemma (2.10)

Let $\vec{a} \in A_N$. Suppose $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$. Then, there are at least $i+1$ cyclic consecutive components of $D^{(n-1)-i}(\vec{a})$ taken from 0 or M such that at least one of them is M, where $M = \max D^k(\vec{a})$.

Proof

Prepare for Theorem 2.12

Remark (2.11)

In Lemma 2.10, we observe that:

(a) $0 \leq i \leq N-1$.

Prepare for Theorem 2.12

Remark (2.11)

In Lemma 2.10, we observe that:

(a) $0 \leq i \leq N-1$.

(b) If $i \leq \min\{n-k-1, N-1\}$, then $D^{(n-1)-i}(\vec{a})$ is in the $(n-k)$-cycle of $\vec{a}$.

Proof

A property about the greatest common divisor of $\vec{a}$ in A_N

Lemma (2.18)

Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$. For all nonnegative integers r, s with $r \leq s$, then $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$.

A property about the greatest common divisor of $\vec{a}$ in A_N

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Let $\vec{a} \in A_N$ *with* $\vec{a} \neq \vec{\mathbf{0}}$. *For all nonnegative integers* r, s *with* $r \leq s$, *then* $\gcd D^r(\vec{a}) \mid \gcd D^s(\vec{a})$. *In particular, we have* $\gcd D^r(\vec{a}) \leq \gcd D^s(\vec{a})$.

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Proof

Example

$$\begin{aligned} \vec{a} &= (3,3,3,3,3,9) & \gcd \vec{a} &= 3 \\ D(\vec{a}) &= (0,0,0,0,6,6) & \gcd D(\vec{a}) &= 6 \\ D^2(\vec{a}) &= (0,0,0,6,0,6) & \gcd D^2(\vec{a}) &= 6 \\ D^3(\vec{a}) &= (0,0,6,6,6,6) & \gcd D^3(\vec{a}) &= 6 \\ &\vdots & &\vdots \end{aligned}$$

The greatest common divisor of $\vec{a}$ in the cycle of $\vec{a}$

Lemma (2.19)

Let $\vec{a} \in A_N$ *with* $\vec{a} \neq \vec{\mathbf{0}}$.

The greatest common divisor of $\vec{a}$ in the cycle of $\vec{a}$

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Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$. Suppose that $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$.

The greatest common divisor of $\vec{a}$ in the cycle of $\vec{a}$

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Proof

The greatest common divisor of $\vec{a}$ in the cycle of $\vec{a}$

Lemma (2.19)

Let $\vec{a} \in A_N$ with $\vec{a} \neq \vec{\mathbf{0}}$. Suppose that $D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$ is the $(n-k)$-cycle of $\vec{a}$. Then, we have $\gcd D^r(\vec{a}) = \gcd D^s(\vec{a})$ for all $k \leq r, s \leq n-1$.

Proof

Example

$$\vec{a} = (0,1,2,2,1,0) \qquad D^3(\vec{a}) = (1,0,1,1,0,1)$$

$$D(\vec{a}) = (1,1,0,1,1,0) \qquad D^4(\vec{a}) = (1,1,0,1,1,0)$$

$$D^2(\vec{a}) = (0,1,1,0,1,1) \qquad = D(\vec{a})$$

The converse of Lemma 2.6 and Lemma 2.19 may not be true even if put them together

Example (2.20)

$$\vec{e}_1 = (1,0,0,0,0,0) \in A_6$$
$$D(\vec{e}_1) = (1,0,0,0,0,1)$$
$$D^2(\vec{e}_1) = (1,0,0,0,1,0)$$
$$D^3(\vec{e}_1) = (1,0,0,1,1,1)$$
$$D^4(\vec{e}_1) = (1,0,1,0,0,0)$$
$$D^5(\vec{e}_1) = (1,1,1,0,0,1)$$
$$D^6(\vec{e}_1) = (0,0,1,0,1,0)$$
$$D^7(\vec{e}_1) = (0,1,1,1,1,0)$$
$$D^8(\vec{e}_1) = (1,0,0,0,1,0) = D^2(\vec{e}_1)$$

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$\Longrightarrow$ the period of $\vec{e}_1$ is $(8-2)=6$

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$$D^8(\vec{e}_1) = (1,0,0,0,1,0) = D^2(\vec{e}_1)$$

$\Longrightarrow$ the period of $\vec{e}_1$ is $(8-2)=6$, and the 6-cycle of $\vec{e}_1$ is $D^2(\vec{e}_1), D^3(\vec{e}_1), D^4(\vec{e}_1), D^5(\vec{e}_1), D^6(\vec{e}_1), D^7(\vec{e}_1)$

The converse of Lemma 2.6 and Lemma 2.19 may not be true even if put them together

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Note that $\gcd D^i(\vec{e}_1) = 1 = \max D^i(\vec{e}_1)$ for all $i = 0, 1, \cdots, 7$

The converse of Lemma 2.6 and Lemma 2.19 may not be true even if put them together

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\vec{e}_1 &= (1,0,0,0,0,0) \in A_6 \\
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\qquad
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D^8(\vec{e}_1) &= (1,0,0,0,1,0) \\
&= D^2(\vec{e}_1)
\end{aligned}$$

$\Longrightarrow$ the period of $\vec{e}_1$ is $(8-2)=6$, and the 6-cycle of $\vec{e}_1$ is $D^2(\vec{e}_1), D^3(\vec{e}_1), D^4(\vec{e}_1), D^5(\vec{e}_1),\ D^6(\vec{e}_1), D^7(\vec{e}_1)$

Note that $\gcd D^i(\vec{e}_1) = 1 = \max D^i(\vec{e}_1)$ for all $i = 0, 1, \cdots, 7$

However, $D^0(\vec{e}_1) = \vec{e}_1, D(\vec{e}_1)$ are not in the cycle of $\vec{e}_1$

A property about the complement of N-tuples in A_N

Lemma (3.4)

Let $\vec{a} = (a_1, a_2, \cdots, a_N) \in A_N$ *and* $\max \vec{a} = M$. *Suppose the cycle of* $\vec{a}$ *is similar to the cycle of* $\vec{b}$, *where* $\vec{b} \in (\mathbb{Z}_2)^6$ *and the period of* $\vec{b}$ *is equal to the period of* $\vec{a}$.

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Proof

A property about the complement of N-tuples in A_N

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Proof

Example

$$\vec{a} = (0, 2, 4, 4, 2, 0) \qquad \vec{a^c} = (4, 2, 0, 0, 2, 4)$$

$$D(\vec{a}) = (2, 2, 0, 2, 2, 0) \qquad D(\vec{a^c}) = (2, 2, 0, 2, 2, 0)$$

$$D^2(\vec{a}) = (0, 2, 2, 0, 2, 2) \qquad D^2(\vec{a^c}) = (0, 2, 2, 0, 2, 2)$$

$$D^3(\vec{a}) = (2, 0, 2, 2, 0, 2) \qquad D^3(\vec{a^c}) = (2, 0, 2, 2, 0, 2)$$

$$D^4(\vec{a}) = (2, 2, 0, 2, 2, 0) = D(\vec{a}) \qquad D^4(\vec{a^c}) = (2, 2, 0, 2, 2, 0) = D(\vec{a^c})$$

A remark about the proof in Lemma 3.4

Remark (3.5)

In the proof of Lemma 3.4, $D^{s+1}(\vec{b})$ is in the cycle of $\vec{b}$.

Proof

Notations

Now, we define $T\colon A_N \to A_N$ by

$$T(x_1, x_2, \cdots, x_{N-1}, x_N) = (x_2, x_3, \cdots, x_N, x_1)$$

for all $(x_1, x_2, \cdots, x_{N-1}, x_N) \in A_N$.

Notations

Now, we define $T : A_N \to A_N$ by

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$\mathscr{D} = D\,|_{(\mathbb{Z}_2)^N}$, $\mathscr{T} = T\,|_{(\mathbb{Z}_2)^N}$, and

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On the other hand, we fix the following notations:
$\mathscr{D} = D\,|_{(\mathbb{Z}_2)^N}$, $\mathscr{T} = T\,|_{(\mathbb{Z}_2)^N}$, and $\mathscr{D}^0 = \mathscr{T}^0 = \mathscr{I}$, where $\mathscr{I}$ is the identity on $(\mathbb{Z}_2)^N$.

A property about the complement of N-tuples in cycles

Lemma (3.6)

Let $\vec{a} = (a_1, a_2, \cdots, a_N) \in A_N$ *and* $\max \vec{a}$*=M. Suppose that*

$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$*-cycle of* $\vec{a}$*.*

A property about the complement of N-tuples in cycles

Lemma (3.6)

Let $\vec{a} = (a_1, a_2, \cdots, a_N) \in A_N$ *and* $\max \vec{a}$=*M. Suppose that*

$$D^k(\vec{a}), D^{k+1}(\vec{a}), \cdots, D^{n-1}(\vec{a})$$

is the $(n-k)$*-cycle of* $\vec{a}$*. Then,* $\vec{a^c} = (M - a_1, M - a_2, \cdots, M - a_N)$ *is in the* $(n-k)$*-cycle of* $\vec{a}$ *if and only if* $\vec{a} = \vec{\mathbf{0}}$*.*

Proof

A property about D and T

Lemma (3.7)

Let $\vec{x}, \vec{y} \in A_N$ and c be a nonnegative integer, then

(a) $T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y})$.

(b) $D \circ T = T \circ D$.

Proof

Prepare for Theorem 3.13

Remark (3.8)

Let $x, y \in \mathbb{Z}_2$. Then, $|x - y| = x + y$.

Proof

Prepare for Theorem 3.13

Remark (3.9)

Let $\mathscr{L} : (\mathbb{Z}_2)^N \to (\mathbb{Z}_2)^N$ be a function.

Prepare for Theorem 3.13

Remark (3.9)

Let $\mathscr{L} : (\mathbb{Z}_2)^N \to (\mathbb{Z}_2)^N$ be a function. Then, we know that $\mathscr{L}$ is a linear transformation if and only if $\mathscr{L}(\vec{x} + \vec{y}) = \mathscr{L}(\vec{x}) + \mathscr{L}(\vec{y})$ for all $\vec{x}, \vec{y} \in (\mathbb{Z}_2)^N$.

Proof

A property about $\mathscr{T}$

Lemma (3.10)

$\mathscr{T}^i$ is a linear transformation for each $i = 0, 1, 2, \cdots$.

Proof

A property about $\mathscr{D}$

Lemma (3.11)

$\mathscr{D}^i$ *is a linear transformation for each* $i = 0, 1, 2, \cdots$.

Proof

Prepare for Theorem 3.13

Lemma (3.12)

Let $\vec{a} \in A_N$. Suppose that r, s, t are nonnegative integers such that $s \leq r$ and $s \leq t$.

Prepare for Theorem 3.13

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Let $\vec{a} \in A_N$. Suppose that r, s, t are nonnegative integers such that $s \leq r$ and $s \leq t$. If $D^r(\vec{a}) = D^s(\vec{a})$, then

$$D^{(r-s)i}(D^t(\vec{a})) = D^t(\vec{a})$$

for each $i = 0, 1, 2, \cdots$.

Proof

Prepare for Theorem 3.15

Lemma (3.14)

Let r, s be nonnegative integers. Then, we have:

(a) $\mathscr{D} = \mathscr{I} + \mathscr{T}$.

Prepare for Theorem 3.15

Lemma (3.14)

Let r, s *be nonnegative integers. Then, we have:*

(a) $\mathscr{D} = \mathscr{I} + \mathscr{T}$.

(b) *If* $2^r \equiv s \pmod{N}$, *then* $\mathscr{D}^{2^r} = \mathscr{I} + \mathscr{T}^s$.

Proof

Similar cycles of 6-tuples in A_6

Theorem (4.3)

Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$.

Similar cycles of 6-tuples in A_6

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(ii) (3-cycle) $(0, 1, 1, 0, 1, 1), (1, 0, 1, 1, 0, 1), (1, 1, 0, 1, 1, 0)$.

Similar cycles of 6-tuples in A_6

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Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$.

(ii) (3-cycle) $(0, 1, 1, 0, 1, 1), (1, 0, 1, 1, 0, 1), (1, 1, 0, 1, 1, 0)$.

(iii) (6-cycle) $(0, 1, 0, 0, 0, 1), (1, 1, 0, 0, 1, 1), (0, 1, 0, 1, 0, 0),$
$(1, 1, 1, 1, 0, 0), (0, 0, 0, 1, 0, 1), (0, 0, 1, 1, 1, 1)$.

Similar cycles of 6-tuples in A_6

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Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0,0,0,0,0,0)$.

(ii) (3-cycle) $(0,1,1,0,1,1),(1,0,1,1,0,1),(1,1,0,1,1,0)$.

(iii) (6-cycle) $(0,1,0,0,0,1),(1,1,0,0,1,1),(0,1,0,1,0,0),$
$(1,1,1,1,0,0),(0,0,0,1,0,1),(0,0,1,1,1,1)$.

(iv) (6-cycle) $(1,0,0,0,1,0),(1,0,0,1,1,1),(1,0,1,0,0,0),$
$(1,1,1,0,0,1),(0,0,1,0,1,0),(0,1,1,1,1,0)$.

Proof

The complement of 6-tuples in $(\mathbb{Z}_2)^6$

Definition (4.6)

Let $\vec{a} = (a_1, a_2, \cdots, a_6) \in (\mathbb{Z}_2)^6$. The *complement of* $\vec{a}$ is defined to be $(1 - a_1, 1 - a_2, \cdots, 1 - a_6)$ and we denote it by $\vec{a^c}$.

The complement of 6-tuples in $(\mathbb{Z}_2)^6$

Definition (4.6)

Let $\vec{a} = (a_1, a_2, \cdots, a_6) \in (\mathbb{Z}_2)^6$. The *complement of* $\vec{a}$ is defined to be $(1 - a_1, 1 - a_2, \cdots, 1 - a_6)$ and we denote it by $\vec{a^c}$.

Remark (4.7)

If $\vec{a} = (a_1, a_2, \cdots, a_6) \in (\mathbb{Z}_2)^6$, then $\vec{a^c} \in (\mathbb{Z}_2)^6$.

Proof

A property of the complement of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.8)

If $\vec{a} = (a_1, a_2, \cdots, a_6) \in (\mathbb{Z}_2)^6$, *then* $\pi * \vec{a^c} = (\pi * \vec{a})^c$ *for all* $\pi \in \mathcal{D}_6$.

Proof

A property of the complement of 6-tuples in $(\mathbb{Z}_2)^6$

Lemma (4.8)

If $\vec{a} = (a_1, a_2, \cdots, a_6) \in (\mathbb{Z}_2)^6$, *then* $\pi * \vec{a^c} = (\pi * \vec{a})^c$ *for all* $\pi \in \mathcal{D}_6$.

Proof

Similar cycles of Diffy Hexagons

Corollary (4.11)

Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0,0,0,0,0,0)$.

Similar cycles of Diffy Hexagons

Corollary (4.11)

Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$.

(ii) (1-cycle) $(0, 1, 1, 0, 1, 1)$.

Similar cycles of Diffy Hexagons

Corollary (4.11)

Let $\vec{a} \in A_6$. Then, the cycle of $\vec{a}$ is similar to one of the following cycles:

(i) (1-cycle) $(0, 0, 0, 0, 0, 0)$.

(ii) (1-cycle) $(0, 1, 1, 0, 1, 1)$.

(iii) (2-cycle) $(0, 0, 1, 0, 1, 0), (0, 1, 1, 1, 1, 0)$.

Proof

The period of Diffy N-gons

Let r, s be positive integers. Suppose that $N = 2^s$ and $\vec{e}_1 = (1, 0, \cdots, 0) \in A_N$.

The period of Diffy N-gons

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The period of Diffy N-gons

If we had enough time, we would like to have further discussions about that at what positive integer N above conclusion holds (even if the identification we use in this chapter is necessary).