# The Study of Secure-dominating Set of Graph Products

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# Outline

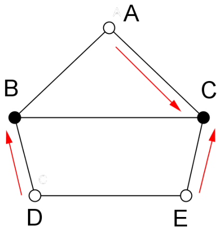
# Secure Set

## An Attack on $S$

$A : S \to P(V(G) - S)$ such that $A(u) \subseteq N_G[u] - S$ for any $u \in S$ and $A(u) \cap A(v) = \emptyset$ for any $u \neq v \in S$.

## A Defense of $S$

$D : S \to P(S)$ such that $D(u) \subseteq N_G[u] \cap S$ for any $u \in S$ and $D(u) \cap D(v) = \emptyset$ for any $u \neq v \in S$.



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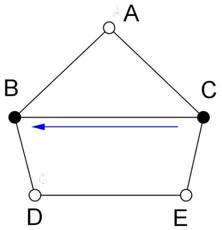

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## R. Brigham, R. Dutton and S. Hedetniemi, 2004

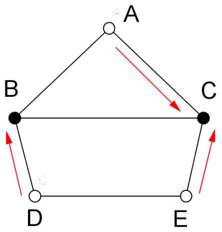
A subset $S$ of $V(G)$ is a secure set of $G$ if for any attack $A$ on $S$, there exists a defense $D$ of $S$ such that $|D(u)| \geq |A(u)|$ for any $u \in S$.

# Secure Set

A subset $S$ of $V(G)$ is a dominating set if $N_G[S] = V(G)$.

## Secure-dominating Set and Secure-dominating Number

- A subset $S$ of $V(G)$ is a secure-dominating set of $G$ if $S$ is a secure set of $G$ that is also a dominating set of $G$.
- The secure-dominating number $\gamma^s(G)$ of $G$ is the minimum cardinality of secure-dominating sets of $G$.



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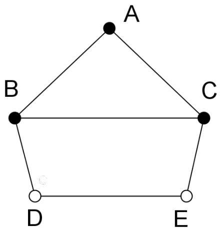
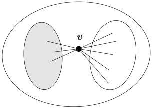


# Preliminary

## Proposition 2.1

If $S$ is a secure set of a graph $G$, then for each vertex $v$ in $S$, $|N_G[v] \cap S| \geq |N_G(v) - S|$.




$|N_G[v] \cap S| < |N_G(v) - S|$

## Theorem 2.4

For any graph $G$, $\gamma^s(G) \geq \lceil \frac{|G|}{2} \rceil$.

Let $S$ be a secure-dominating set.

$$|S| \geq \sum_{u \in S} |D(u)| \geq \sum_{u \in S} |A(u)| \geq |V(G) - S|.$$

# Known Results

## C.-L. Chang, T.-P. Chang and D. Kuo, 2009

$$\begin{cases} \gamma^s(P_m \Box P_n) = \lceil \frac{mn}{2} \rceil, & \text{if } m \text{ and } n \text{ are at least two;} \\ \gamma^s(P_m \Box C_n) = \lceil \frac{mn}{2} \rceil, & \text{if } m \geq 2 \text{ and } n \geq 3; \\ \gamma^s(C_m \Box C_n) = \frac{mn}{2} + 1, & \text{if } m \equiv 2 \pmod 4 \\ & \text{and } n \equiv 3 \pmod 4; \\ \gamma^s(C_m \Box C_n) = \lceil \frac{mn}{2} \rceil, & \text{if } m \not\equiv 2 \pmod 4 \text{ or } n \not\equiv 3 \pmod 4. \end{cases}$$

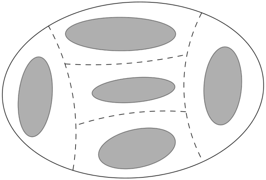
$\gamma^s(K_{m_1, m_2, \ldots, m_l}) = \lceil \frac{m_1 + m_2 + \cdots + m_l}{2} \rceil$, if $l \geq 2$.

## K.-P. Huang and S.-T. Juan, 2011

If $l$ is an integer at least 2 and $m_1, m_2, \ldots, m_l$ are positive integers, then $\gamma^S(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_l}) = \gamma^S(K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_l}) = \lceil \frac{m_1 \times m_2 \times \cdots \times m_l}{2} \rceil$.

# Main Idea 1

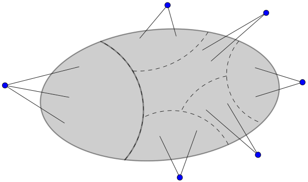
Let $V_1, V_2, \ldots, V_k$ be a partition of $V(G)$.
If $S_i$ is a secure-dominating set of $G[V_i]$ for each $1 \leq i \leq k$,
and $N[S_i] \cap N[S_j] = \emptyset$, for any $1 \leq i \neq j \leq k$,
then $S_1 \cup S_2 \cup \cdots \cup S_k$ is a secure-dominating set of $G$.



If $u$ is not in $S$, then $u \in V_i - S$ for some $i$.
Only the vertices in $S_i$ can be attacked by $u$.

# Main Idea 2

Let $S$ be a dominating set and $V(G) - S = \{v_1, v_2, \ldots, v_k\}$.
If $S$ can be partitioned into $S_1, S_2, \ldots, S_k$ such that,
for each $i$, $N(v_i) \cap S \subseteq N[S_i] \cap S$.
Then $S$ is a secure-dominating set.

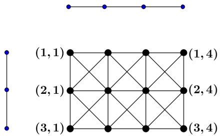


If some vertex in $S$ is attacked by $v_i$, then we can use some vertex in $S_i$ to defense the attack.

# Results on the strong product of two graphs

## Strong Product of Graphs

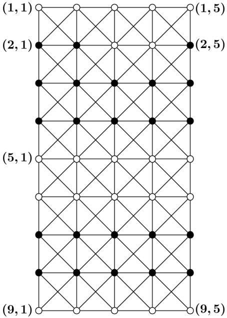
Let $X$ and $Y$ be two graphs.
The strong product $X \boxtimes Y$ of $X$ and $Y$ is the graph such that
$V(X \boxtimes Y) = V(X) \times V(Y)$,
$(x_1, y_1) \sim (x_2, y_2)$ in $X \boxtimes Y$ if and only if
$x_1 = x_2$ or $x_1 \sim x_2$ in $X$, and $y_1 = y_2$ or $y_1 \sim y_2$ in $Y$.




$P_3 \boxtimes P_4$

# $P_9 \boxtimes P_5$

$S_{9,5} = \{(2, j) : j \equiv 1, 2 \pmod 4\} \cup \{(i, j) : i \equiv 0, 3 \pmod 4, 1 \le j \le 5\}$.

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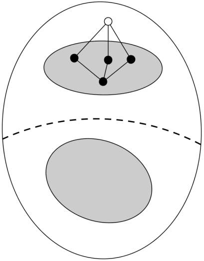

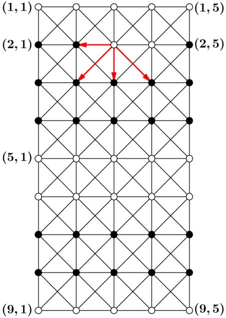


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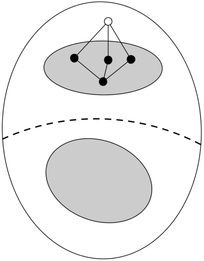

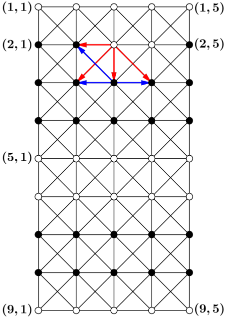


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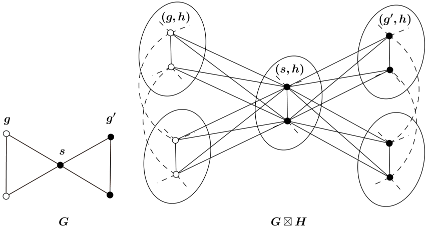
# Results on the strong product of two graphs

**Lemma**

Let $G$ and $H$ are two graphs. If $S_G$ is a secure-dominating set of $G$, then $S = \{(s, h) : s \in S_G, h \in V(H)\}$ is a secure-dominating set of $G \boxtimes H$.



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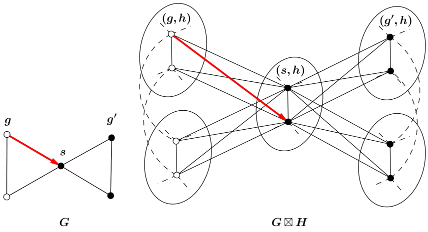

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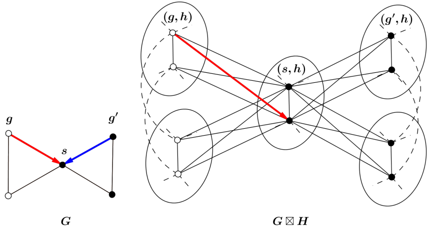

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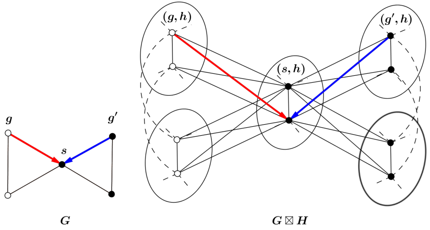


# Results on the strong product of two graphs

**Theorem 6.2**

Let $G$ and $H$ be two gaphs. We have $\gamma^s(G \boxtimes H) \leq \min\{\gamma^s(G)|H|, |G|\gamma^s(H)\}$.

Let $S_G$ be a secure-dominating set of $G$ with size $\gamma^s(G)$ and $S_H$ be a secure-dominating set of $H$ with size $\gamma^s(H)$.
By Lemma, $S = \{(s,h) : s \in S_G, h \in V(H)\}$ and $S' = \{(s',h') : s' \in V(G), h' \in S_H\}$ are both secure-dominating sets of $G \boxtimes H$.
Hence, $\gamma^s(G \boxtimes H) \leq \min\{|S|, |S'|\} = \min\{|S_G||H|, |G||S_H|\} = \min\{\gamma^s(G)|H|, |G|\gamma^s(H)\}$.

# Results on the strong product of two graphs

**Corollary 6.3**

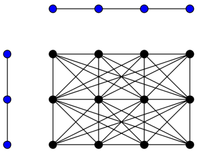
Let $G$ and $H$ be two graphs. If $\gamma^s(G) = \frac{|G|}{2}$, then $\gamma^s(G \boxtimes H) = \frac{|G \boxtimes H|}{2}$.

If $\gamma^s(G) = \frac{|G|}{2}$, then $\gamma^s(G \boxtimes H) \leq min\{\gamma^s(G)|H|, |G|\gamma^s(H)\} = min\{\frac{|G||H|}{2}, |G|\gamma^s(H)\} = \frac{|G||H|}{2} = \frac{|G \boxtimes H|}{2}$ by Theorem 2.4. By Theorem 6.2, $\gamma^s(G \boxtimes H) \geq \lceil \frac{|G \boxtimes H|}{2} \rceil$. Hence, $\gamma^s(G \boxtimes H) = \frac{|G \boxtimes H|}{2}$.

# Results on $H[G]$

## Lexicographic Product of Graphs

Let $X$ and $Y$ be two graphs.
The lexicographic product $X[Y]$ of $X$ and $Y$ is the graph such that $V(X[Y]) = V(X) \times V(Y)$,
$(x_1, y_1) \sim (x_2, y_2)$ in $X[Y]$ if and only if
either $x_1 \sim x_2$ in $X$, or $x_1 = x_2$ and $y_1 \sim y_2$ in $Y$.



$P_3[P_4]$

# Results on $H[G]$

## Lemma 7.1

Let $G$ and $H$ be two graphs.
If $S_G$ is a secure-dominating set of $G$,
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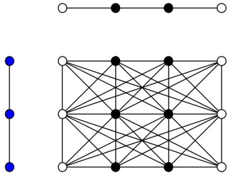

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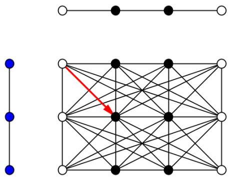

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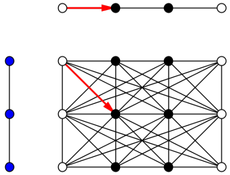

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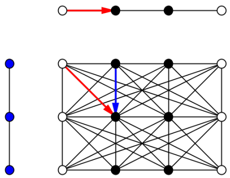

$P_3[P_4]$

# Results

## Theorem 7.2

Let $G$ and $H$ be two graphs,
we have $\gamma^s(H[G]) \leq \gamma^s(G)|H|$.

Let $S_G$ is a secure-dominating set with size $\gamma^s(G)$,
then $\gamma^s(H[G]) \leq |H[S_G]| \leq \gamma^s(G)|H|$.

# Conclusions

$$
\begin{cases}
\gamma^s(P_m \boxtimes P_n) = \lceil \frac{mn}{2} \rceil; & \\
\gamma^s(P_m \boxtimes C_n) = \lceil \frac{mn}{2} \rceil, & \text{if } m \text{ is even or } n \not\equiv 2 \pmod 4; \\
\lceil \frac{mn}{2} \rceil \le \gamma^s(P_m \boxtimes C_n) \le \lceil \frac{mn}{2} \rceil + 1, & \text{if } m \text{ is odd and } n \equiv 2 \pmod 4; \\
\gamma^s(C_m \boxtimes C_n) = \lceil \frac{mn}{2} \rceil, & \text{if } m \equiv 0 \pmod 4, \text{ or } m \text{ and } n \text{ are both odd except } m \equiv n \equiv 3 \pmod 4; \\
\lceil \frac{mn}{2} \rceil \le \gamma^s(C_m \boxtimes C_n) \le \lceil \frac{mn}{2} \rceil + 1, & \text{if } m \equiv n \equiv 3 \pmod 4 \text{ except } m = n = 7, \text{ or } m \equiv 2 \pmod 4 \text{ and } n \text{ is odd}; \\
\lceil \frac{mn}{2} \rceil \le \gamma^s(C_m \boxtimes C_n) \le \lceil \frac{mn}{2} \rceil + 2, & \text{if } m \text{ and } n \text{ are both 7 or } m \equiv n \equiv 2 \pmod 4.
\end{cases}
$$

# Conclusions

$$\begin{cases} \gamma^s(G \boxtimes H) \le \min\{\gamma^s(G)|H|, |G|\gamma^s(H)\}, & \text{if } G \text{ and } H \text{ are two graphs;} \\ \gamma^s(P_m \boxtimes G) = \gamma^s(K_m \boxtimes G) = \frac{m|G|}{2}, & \text{if } m \text{ is even;} \\ \lceil \frac{m|G|}{2} \rceil \le \gamma^s(P_m \boxtimes G) \le \gamma^s(G) + \frac{(m-1)|G|}{2}, & \text{if } m \text{ is odd;} \\ \lceil \frac{m|G|}{2} \rceil \le \gamma^s(K_m \boxtimes G) \le \gamma^S(G) + \frac{(m-1)|G|}{2}, & \text{if } m \text{ is odd.} \end{cases}$$

$\gamma^s(H[G]) \le \gamma^s(G)|H|$ for any two graphs $G$ and $H$.

# Bibliography


1. R. Aharoni, E. Milner and K. Prikry, Unfriendly partitions of a graph, J. Combin. Theory Ser. B 50 (1990), 1-10.
2. R. Brigham, R. Dutton and S. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$, Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing. Congr. Numer. 167 (2004), 57-63.
3. R. Brigham, R. Dutton and S. Hedetniemi, Security in graphs, Discrete Appl. Math. 155 (2007), 1708-1714.
4. R. Brigham, R. Dutton, T. Haynes and S. Hedetniemi, Powerful alliances in graphs, Discrete Math. 309 (2009), 2140-2147.

# Bibliography


5. C.-L. Chang, T.-P. Chang and D. Kuo, Secure and secure-dominating set of graphs, manuscript (2009).
6. Y.-H. Chang, Secure and secure-dominating sets of graphs, master's dissertation, National Dong Hwa University, Hualien, Taiwan (2009).
7. K.-P. Huang, Secure and Secure-dominating set of Cartesian product graphs, master's dissertation, National Chai Nan University, Nantou, Taiwan (2011).
8. P. Kristiansen, S. Hedetniemi and S. Hedetniemi, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004), 157-177.
9. D. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, NJ 1996.