The Minimum Rank of Buds

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Introduction

All graphs considered in this thesis are simple and connected graphs. For a graph G of order n, we use $E(G)$ as its edge set and $V(G)$ as its vertex set, usually $V(G) = [n] = \{1, 2, ..., n\}$. For an $n \times n$ real symmetric matrix A, $\Gamma(A)$ represents the graph such that $ij \in E(\Gamma(A))$ if and only if the ij-th entry of A is not zero, indicating that the matrix A is **associated** with $\Gamma(A)$.

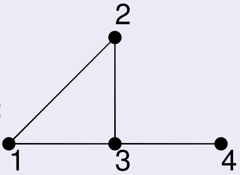
Matrices associated with graph G

Example

The 4×4 matrix A is associated with $\Gamma(A)$.

$$A = \begin{bmatrix} 1 & 1/5 & -1 & 0 \\ 1/5 & 0 & 1 & 0 \\ -1 & 1 & -4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$\Gamma(A)$:



Note that the diagonal entries do not need to be 0.

The **minimum rank** of a graph G, denoted by $m(G)$, is defined to be the integer

$$m(G) = \min\{\text{rank}(A) : \Gamma(A) = G\},$$

where the minimum is taking over all $n \times n$ symmetric matrices A. The minimum rank of G is related to the **maximum nullity** of G, denoted by

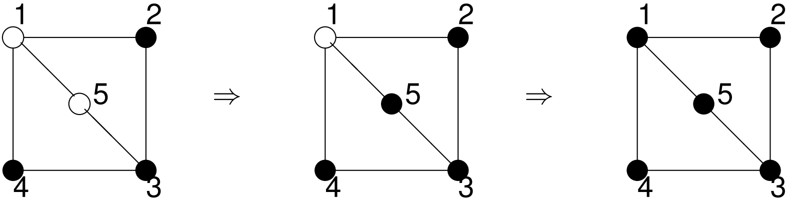
$$M(G) = \max\{\text{nullity}(A) : \Gamma(A) = G\}.$$

It is well-known that $m(G) + M(G) = n$ for all graphs.

Color-change rule

1. **Initial configuration**
2. **Color-change rule**
3. **Zero-forcing set**

The minimum size of a zero-forcing set of G is denoted by $Z(G)$.



Notation

Let G be a simple connected graph, $x, y \in V(G)$, $i, j \in \mathbb{N}$, and a column vector $w \in \mathbb{R}^n$. We use the following notations:

1. $x \sim y$ means that x is a neighbor of y.
2. $G_1(x)$ denotes the set of the neighbors of vertex $x \in G$.
3. $|G|$ denotes the order of graph G.
4. $[i, j] = [j] - [i-1] = \{i, i+1, \cdots, j-1, j\}$.
5. supp(w):$= \{i \in \mathbb{N} |$ the i-th entry of w is not zero $\}$.

Known Result

Lemma

If H is an induced subgraph of G, then we have $m(H) \leq m(G)$.

Theorem (M. Fiedler 1969)

Let A be a symmetric matrix of order n. Then the following (i)-(ii) are equivalent.
(i) $\text{rank}(A+D) \geq n-1$ for any diagonal matrix D.
(ii) $\Gamma(A) = P_n$. □

Example

The following matrix P satisfies $\Gamma(P) = P_n$, and $\text{rank}(P) = n - 1$.

$$P = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ 0 & & & 1 & 1 \end{bmatrix}$$

On the aforementioned Theorem and Example, we know that P_n is the unique graph with minimum rank $n - 1$ among all graphs of order n.

Here we can determine the minimum rank of the graphs with order n, which has an induced subgraph P_{n-1}.

Lemma

If a graph G of order n is not a path and contains an induced subgraph P_{n-1}, then $m(G) = n - 2$.

Lemma

The minimum rank of C_n is $n - 2$.

Example

The matrix $A_t = (a_{ij})$ defined as follows satisfies $\Gamma(A_t) = C_t$ with $\text{rank}(A_t) = t - 2$.

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \text{ and } i, j \notin \{1, t-1, t\}; \\ 1, & \text{if } i = j, i, j \in \{1, t-1\}; \\ t-2, & \text{if } i = j = t; \\ 1, & \text{if } |i - j| = 1; \\ (-1)^{t-1}, & \text{if } (i, j) = (1, t) \text{ or } (i, j) = (t, 1); \\ 0, & \text{otherwise.} \end{cases}$$

$$A_t = \begin{bmatrix} 1 & 1 & & & & (-1)^{t-1} \\ 1 & 2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & 2 & 1 & \\ & & & 1 & 1 & 1 \\ (-1)^{t-1} & & & & 1 & t-2 \end{bmatrix} \quad (*)$$

Proposition

For a cycle C_n, the set of any two adjacent vertices is a zero-forcing set.

Proposition (AIM MinimumRank-Special Graphs Work Group 2008)

Let G be any graph. Then $M(G) \leq Z(G)$. □

Theorem (AIM MinimumRank-Special Graphs Work Group 2008)

For each of the following families of graphs, $Z(G) = M(G)$

1. Any graph G such that $|G| \leq 6$.
2. K_n, P_n, C_n.
3. Any tree T.
4. All the graphs listed in Table 1. □

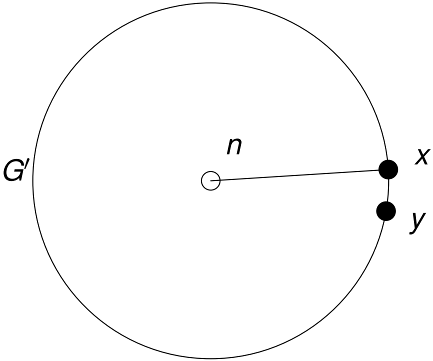
Table1

Result #	G	Order	$M(G)$	$\mathrm{mr}(G)$
3.1	Q_n (hypercube)	2^n	2^{n-1}	2^{n-1}
3.2	T_n (supertriangle)	$\frac{1}{2}n(n+1)$	n	$\frac{1}{2}n(n-1)$
3.3	$K_s \square P_t$	st	s	$s(t-1)$
3.7	$P_s \square P_t$	st	$\min\{s,t\}$	$st-\min\{s,t\}$
3.13	$P_s \boxtimes P_t$	st	$s+t-1$	$(s-1)(t-1)$
3.8	$C_s \square P_t$	st	$\min\{s,2t\}$	$st-\min\{s,2t\}$
3.9	Möbius ladder	$2n$	4	$2n-4$
3.11	$K_s \square K_t$	st	$st-s-t+2$	$s+t-2$
3.12	$C_s \square K_t, s \geqslant 4$	st	$2t$	$(s-2)t$
3.14	$K_t \circ K_s, t \geqslant 2$	$st+t$	$st-1$	$t+1$
3.15	$\overline{C_n}, n \geqslant 5$	n	$n-3$	3
3.17	$\overline{T}$, T a tree (with $\|T\|=n$), $n \geqslant 4, T \neq K_{1,n-1}$	n	$n-3$	3
3.18	$L(K_n)$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}(n^2-3n+4)$	$n-2$
3.20 3.21	$L(G)$ (with $\|G\|=n$) if G has a Hamiltonian path or contains $K_{k,n-k}$ as a subgraph $(1<k<n-1)$			$n-2$
3.24	$L(T)$, T a tree and $\ell =$ # pendent vertices of T	$\|T\|-1$	$\ell-1$	$\|T\|-\ell$
3.26	Petersen	10	5	5
3.28	4-Antiprism	8	4	4

Add a vertex n to C_{n-1}

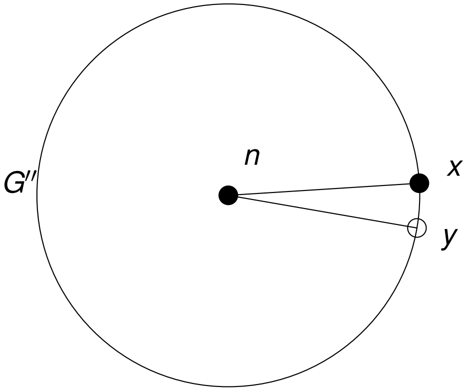
Proposition

Suppose that there is exactly one edge which joins n to some vertex $x \in C_{n-1}$. Then the minimum rank of the new graph G' is $n-2$, and $M(G') = Z(G') = 2$.

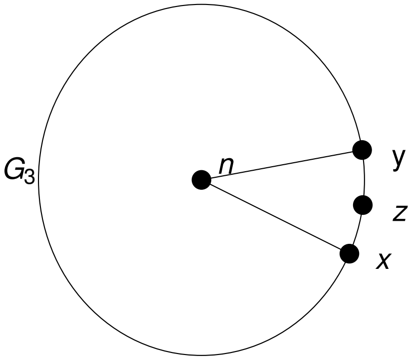


Proposition

Let $x, y \in V(C_{n-1})$ and $x \sim y$. If there are exactly two edges incident on n such that $n \sim x$ and $n \sim y$, then the minimum rank of this new graph G'' is $n-2$, and $M(G'') = Z(G'') = 2$.



Let G_3 be a graph of order n. G_3 is obtained by adding a vertex n and two edges to a cycle C_{n-1}, and the vertex n is adjacent to two vertices which have distance 2.



Before we discuss the situations of G_3, we define four types $n \times n$ matrices W, X, Y, Z as follows.

1. When $n = 4k + 1, k \in \mathbb{N}$,

$$w_{ij} = \begin{cases} 1, & \text{if one of } i, j \text{ is } 1\text{, and the other is } n-1 \text{ or } n\,; \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3)\,; \\ 1, & \text{if } |i-j| = 1, \forall i,j < n\,; \\ 0, & \text{otherwise.} \end{cases}$$

$$W = \begin{bmatrix} 0 & 1 & & & 1 & 1 \\ 1 & 0 & 1 & & & 0 \\ & 1 & \ddots & \ddots & & 1 \\ & & \ddots & \ddots & 1 & \\ 1 & & & 1 & 0 & 0 \\ 1 & 0 & 1 & & 0 & 0 \end{bmatrix}.$$

2. When $n = 4k + 2, k \in \mathbb{N}$,

$$x_{ij} = \begin{cases} 1, & \text{if one of } i, j \text{ is 1, and the other is } n-1 \text{ or } n \text{ ;} \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \text{ ;} \\ 1, & \text{if } |i-j| = 1, \forall i,j < n-1 \text{ ;} \\ 1, & \text{if } i = j, i,j \in \{1, n-2, n-1\} \text{ ;} \\ -1, & (i,j) = (n-2, n-1) \text{ or } (i,j) = (n-1, n-2) \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = \begin{bmatrix} 1 & 1 & & & & 1 & 1 \\ 1 & 0 & 1 & & & & 0 \\ & 1 & \ddots & \ddots & & & 1 \\ & & \ddots & 0 & 1 & & \\ & & & 1 & 1 & -1 & \\ 1 & & & & -1 & 1 & 0 \\ 1 & 0 & 1 & & & 0 & 0 \end{bmatrix}.$$

3. When $n = 4k + 3, k \in \mathbb{N}$,

$$y_{ij} = \begin{cases} 1, & \text{if one of } i, j \text{ is 1, and the other is } n-1 \text{ or } n \text{ ;} \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \text{ ;} \\ 1, & \text{if } |i-j| = 1, \forall i, j < n-1 \text{ ;} \\ -1, & (i,j) = (n-2, n-1) \text{ or } (i,j) = (n-1, n-2) \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

$$Y = \begin{bmatrix} 0 & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 0 \\ & 1 & \ddots & \ddots & & & 1 \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & & -1 & \\ 1 & & & & -1 & 0 & 0 \\ 1 & 0 & 1 & & & 0 & 0 \end{bmatrix}.$$

4. When $n = 4k + 4, k \in \mathbb{N}$,

$$
z_{ij} = \begin{cases} 1, & \text{if one of } i, j \text{ is } 1 \text{, and the other is } n-1 \text{ or } n \text{ ;} \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \text{ ;} \\ 1, & \text{if } |i-j| = 1, \forall i,j < n-1 \text{ except } i+j = 9 \text{ ;} \\ 1, & \text{if } i = j, i,j \in \{1, 4, n-1\} \text{ ;} \\ -1, & (i,j) = (n-2, n-1) \text{ or } (i,j) = (n-1, n-2) \text{ ;} \\ -1, & (i,j) = (4,5) \text{ or } (i,j) = (5,4) \text{ ;} \\ 0, & \text{otherwise.} \end{cases}
$$

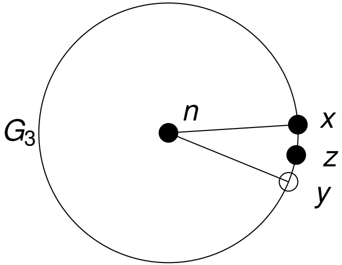
$$
Z = \begin{bmatrix} 1 & 1 & & & & & & & 1 & 1 \\ 1 & 0 & 1 & & & & & & & 0 \\ & 1 & 0 & 1 & & & & & & 1 \\ & & 1 & 1 & -1 & & & & & \\ & & & -1 & 0 & 1 & & & & \\ & & & & 1 & \ddots & \ddots & & & \\ & & & & & \ddots & & 1 & & \\ & & & & & & 1 & 0 & -1 & \\ 1 & & & & & & & -1 & 1 & 0 \\ 1 & 0 & 1 & & & & & & 0 & 0 \end{bmatrix}.
$$

Lemma

The rank of the above-mentioned four $n \times n$ matrices W, X, Y, Z are at most $n - 3$.

Theorem

The minimum rank of the graph G_3 is $n - 3$, and $M(G_3) = Z(G_3) = 3$.



Lemma

For all $n \in \mathbb{N}$, let A_n be the $n \times n$ symmetric matrix defined as follows.

$$A_n = \begin{bmatrix} 1 & 1 & & & & (-1)^{n-1} \\ 1 & 2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & 2 & 1 & \\ & & & 1 & 1 & 1 \\ (-1)^{n-1} & & & & 1 & n-2 \end{bmatrix} \qquad (*)$$

Thus for any subset $S \subseteq [n]$ such that $|S| > 2$, there exists a vector u such that $\mathrm{supp}(u) \subseteq [\max(S) - 1]$ and $\mathrm{supp}(A_n u) = S$.

Now suppose $S = \{t_1, t_2, ..., t_k\} \subseteq [n], k \geq 3$, and $t_1 < t_2 < \cdots < t_k$.

$$A_n \begin{bmatrix} u_1 \\ \vdots \\ u_{t_k - 1} \\ o \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \text{ where } x_i = \begin{cases} \neq 0, & i \in S, \\ = 0, & \text{otherwise.} \end{cases}$$

Theorem

If G is a graph of order n which are obtained by adding a vertex n and at least three edges to a cycle C_{n-1}, then the minimum rank of G is $n-3$, and $M(G) = Z(G) = 3$.

Proof.

Let A_{n-1} be the $(n-1) \times (n-1)$ matrix defined as the matrix $(*)$. We can choose a vector $u \in \mathbb{R}^{n-1}$ such that $\text{supp}(A_{n-1}u) = G_1(n)$. Thus the following matrix B satisfies $\text{rank}(B) = n-3$ and $\Gamma(B) = G$.

$$B = \begin{bmatrix} A_{n-1} & A_{n-1}u \\ u^T A_{n-1} & u^T A_{n-1} u \end{bmatrix}_{n \times n}$$

□

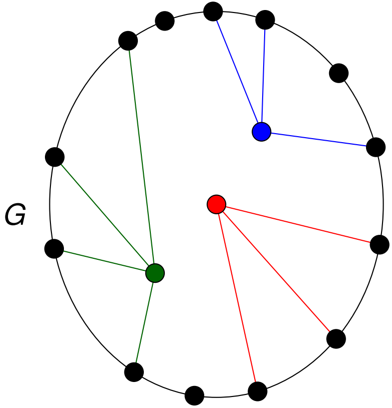
Buds

Definition

For integers $m < n$, let $B_{n,m}$ be a class of graphs G with vertex set $V(G) = [n]$ satisfying the following axioms:

1. The subgraph of G induced on $[n-m]$ is a cycle C_{n-m}, and the subgraph induced on $[n] \setminus [n-m]$ has no edge.
2. Let $1 = t_0 < t_1 < t_2 < \cdots < t_m = n - m + 1$, and $t_j - t_{j-1} > 2$, for all $j \in [m]$. Let $S_i = G_1(n - m + i)$, where $i \in [m]$. Then $|S_i| \geq 3$ and $S_i \subseteq [t_{i-1}, t_i - 1]$.

The graph $G \in B_{n,m}$ is called a Bud based on $[n-m]$.



$G \in B_{16,3}$; G is based on [13].

Theorem

If $G \in B_{n,m}$, then $m(G) = n - m - 2$.

proof

Let $C = [u_1 u_2 \cdots u_m]$, then

$$B = \begin{bmatrix} A & AC \\ C^T A & C^T AC \end{bmatrix}.$$

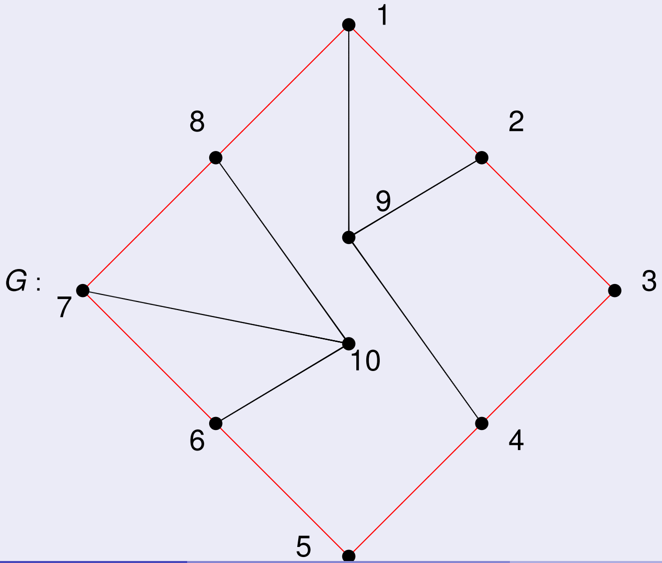
Hence rank(B) =rank$(B[[n-m]|[n]]) = n - m - 2$.

Corollary

If $G \in B_{n,m}$, then $M(G) = Z(G) = m + 2$.

exam

Let G be a graph in $B_{10,2}$ base on [8] such that $G_1(9) = \{1, 2, 4\}$, $G_1(10) = \{6, 7, 8\}$ as in the following figure.



Here we precisely give a matrix B associated with G and the rank of B is 6. Let $S_1 = \{1,2,4\}$, $S_2 = \{6,7,8\}$ and A_8 be the matrix defined in $(*)$. Choose $u_1 = (0,2,-1,0,0,0,0,0)^T$, $u_2 = (-1,1,-1,1,-1,1,0,0)^T$, $C = [u_1 u_2]$. Then the following matrix B is associated with G and rank$(B) = 6$.

$$B = \left[\begin{array}{c|c} A_8 & A_8 C \\ \hline C^T A_8 & C^T A_8 C \end{array}\right] = \left[\begin{array}{cccccccc|cc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 6 & 0 & 1 \\ \hline 2 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right].$$

Conjecture

Conjecture

Let $x, y \in V(C_{n-1})$ with $x \nsim y$. If there are exactly two edges incident on n such that $n \sim x$ and $n \sim y$, then the minimum rank of this new graph is $n-3$, and the maximum nullity and the minimum size of zero-forcing set are equal to 3.

Conjecture

If G is a graph obtained by adding a vertex and some edges to a cycle C_{n-1}, then the maximum nullity of G is equal to the minimum size of a zero-forcing set of G.

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