The Decycling Number on Graphs and Digraphs

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Outline

1. Decycling Problem and Applications
2. Decycling Number and Cycle Packing Number
3. Decycling Number on Graphs
    - Path Product
4. Decycling Number on Digraphs
    - Generalized Kautz Digraph
    - Generalized de Bruijn Digraphs
5. Objective Work

Definition and Applications

Definition (Decycling Problem)

Given a directed/undirected graph $G = (V, E)$*, find a minimum set* $D \subset V$ *such that* $G \setminus D$ *is acyclic.*

- Has applications in
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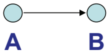

Process A is waiting for the resource on Process B so it can't release its own resource.

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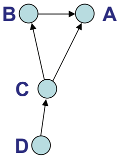

Process A is waiting for no resource so it can release its resource.

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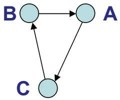

Deadlock:
Competing actions are each waiting for the other to finish, and thus neither ever does.
Solution:
Remove some processes to **break such cycles** and put them in a waiting queue.

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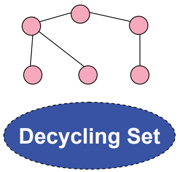
    Example: A 4-regular graph (such as toroidal mesh network).



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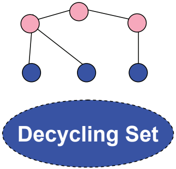

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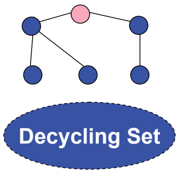

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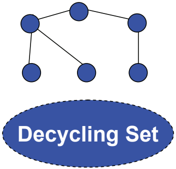

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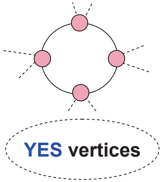

NO vertices on the cycle remain NO.
Failed!

Applications

- Constraint satisfaction problem (Dechter 1990).
- Bayesian inference in artificial intelligence (Bar-Yehuda et al. 1998).
- Converters' placement problem in optical networks (Kleinberg and Kumar 1999).
- VLSI chip design (Festa et al. 2000).

Complexity

- Has been extensively studied.
  - NP-hard
    Reduction from VERTEX COVER (R. Karp 1972).
    Even for planar graphs, bipartite graphs.

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  - Maximum induced forest problem (Erdös, Saks and Sós 1986 Maximum induced trees in graphs).

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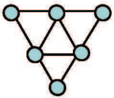
  - Graph Bipartization Problem
    Find $D \subset E$ such that $G \setminus D$ has no odd cycle.
  - Weighted Decycling Problem
    Seek a minimum-weight decycling set D where each vertex has a weight.
  - Loop Cut Set Problem
    Given $A(C) \subseteq V(C)$ for each cycle C of G, find a minimum set D such that $D \cap A(C) \neq \emptyset$.

Relations with Cycle Packing Number

- Is often compared with the following graph parameter.
  - Cycle Packing Number $\nu(G)$: the maximum number of vertex-disjoint cycles of G.



$$\nu(G) = 1$$

  - **Definition:** $\nabla(G) :=$ the decycling number (minimum size of decycling set) of G.

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- **Definition:** $\nabla(k) := \max\{\nabla(G) : \nu(G) = k\}$.
- $\nabla(1) \leq 3$; $\nu(K_5) = 1$ and $\nabla(K_5) = 3$ (Bollob*á*s 1964).
- $\nabla(2) = 6$ and $9 \leq \nabla(3) \leq 12$ (Voss 1968).

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- $\nabla(2) = 6$ and $9 \leq \nabla(3) \leq 12$ (Voss 1968).
- $c_1 k \log k \leq \nabla(k) \leq c_2 k \log k$ for some constants c_1 and c_2 (Erd*ö*s and P*ó*sa 1964).

Relations with Cycle Packing Number

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Theorem (Chen, Fu and Shih 2010, TJM)

For every planar graph G, $\nabla(G) \leq 3\nu(G)$.

Decycling number of outerplanar graphs

Consider outerplanar graphs:

Theorem (Kloks, Lee and Liu 2002)

For every outerplanar graph G, $\nabla(G) \leq 2\nu(G)$.

- An outerplanar graph G is called *lower-extremal* if $\nabla(G) = \nu(G)$ and *upper-extremal* if $\nabla(G) = 2\nu(G)$.

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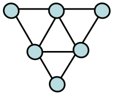
- An outerplanar graph G is called *lower-extremal* if $\nabla(G) = \nu(G)$ and *upper-extremal* if $\nabla(G) = 2\nu(G)$.
- Upper-Extremal Results:
  - We define a *sun graph* S_3 as follows.



  - $\nabla(S_3) = 2 = 2\nu(S_3)$.

Decycling number of outerplanar graphs

Theorem (Chang, Fu, Lien, 2011, JCO)

An outerplanar graph G is upper-extremal if and only if G is an S_3-tree.

- A graph is an *S_3-tree of order t* if it has exactly t vertex-disjoint S_3-subdivisions and every edge not on these S_3-subdivisions belongs to no cycle.

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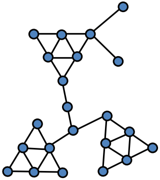
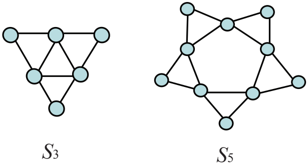
- Example:



An S_3-tree G of order 3, where $\nabla(G) = 6 = 2\nu(G)$.

Decycling number of outerplanar graphs

- Lower-Extremal Results:
  - The following graphs are NOT lower-extremal ($\nabla(G) \neq \nu(G)$):
    Sun graphs S_k with odd number k:



  - $\nabla(S_k) = \lceil \frac{k}{2} \rceil$ and $\nu(S_k) = \lfloor \frac{k}{2} \rfloor$.

Decycling number of outerplanar graphs

Theorem (Chang, Fu, Lien, 2011, JCO)

For an outerplanar graph G, if G has no S_k-subdivision for all odd number k, then G is lower-extremal.

Decycling number of Graphs

Lower Bound of Undirected Graphs

Lemma (Beineke, 1997, JGT)

If G is a connected graph with p vertices ($p > 2$), q edges, and maximum degree Δ, then $\nabla(G) \geq \frac{q-p+1}{\Delta-1}$.

Cartesian Product

Definition ($G\Box H$)

$V(G\Box H) = \{(u,v) | u \in V(G)$ and $v \in V(H)\}$
$E(G\Box H) = \{(u,v)(u',v') | u = u'$ and $(v,v') \in E(H)$ or $(u,u') \in E(G)$ and $v = v'\}$

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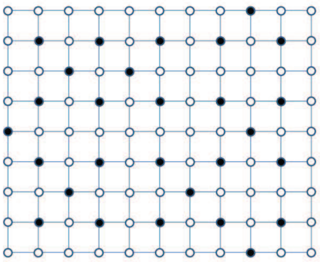
Figure : $P_9\Box P_{11}$



Decycling number of $C_m \square C_n$

Theorem (Pike, Zou, 2005, SIDMA)

$$\nabla(C_m \square C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \textit{if } m = 4, \\ \lceil \frac{3m}{2} \rceil & \textit{if } n = 4, \\ \lceil \frac{mn+2}{3} \rceil & \textit{otherwise} . \end{cases}$$

Decycling number of Path Product $P_m \square P_n$

Lower bound

Theorem (Luccio, 1998, IPL)

If $m, n \geq 2$, *then* $\nabla(P_m \square P_n) \geq \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Decycling number of Path Product $P_m \Box P_n$

Theorem (Madelaine and Stewart, 2008, DISC)

Table :

$m \backslash n$	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

: increasing the lower bound in this paper

: decreasing the upper bound in this paper

In Table, A: $\nabla(P_m \Box P_n) = F_{m,n}$, *B:* $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$, *C:* $\nabla(P_m \Box P_n) \leq F_{m,n} + 2$, *where* $F_{m,n} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Decycling number of Path Product $P_m \square P_n$

New Lower bound

Proposition (Observation)

If $m \geq 5$ *and* $f_{m,n} = \frac{(m-1)(n-1)+1}{3}$ *is an integer, then each decycling set* S *of size* $f_{m,n}$ *satisfies the following two properties:*
(1) S *contains exactly one vertex of degree* 3 *and contains no vertex of degree 2; and*
(2) S *induces a subgraph of* $P_m \square P_n$ *with no edges.*

Decycling number of Path Product $P_m \Box P_n$

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(2) S induces a subgraph of $P_m \Box P_n$ with no edges.

Theorem (Lien, Fu, Shih, 2014, DMAA)

If $m \geq 5$, mn is even and $f_{m,n}$ is an integer, then
$\nabla(P_m \Box P_n) \geq f_{m,n} + 1 = F_{m,n} + 1$,*where* $f_{m,n} = \frac{(m-1)(n-1)+1}{3}$.

Decycling number of Path Product $P_m \square P_n$

Proof.

Decycling number of Path Product $P_m \Box P_n$

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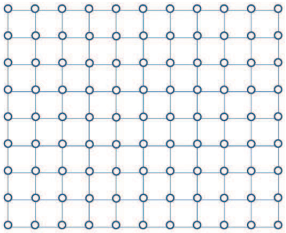
By above Proposition, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$.



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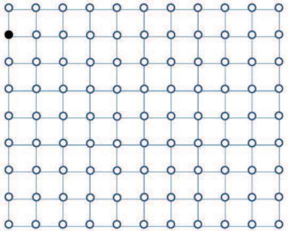

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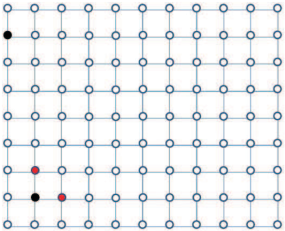

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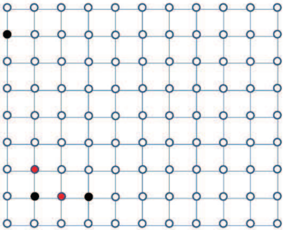

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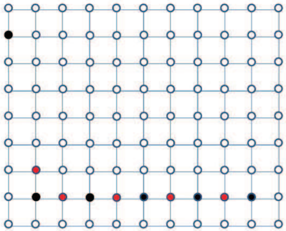

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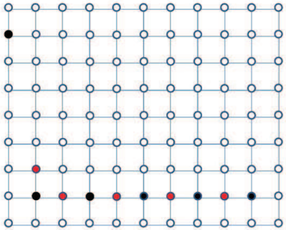

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$v_{m-1,2}, v_{m-1,4}, \cdots, v_{m-1,n-1} \in S$.
Hence, $n-1$ is even and n is odd.

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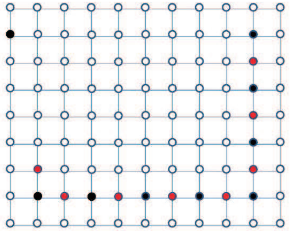

Similarly,
$v_{m-3,n-1}, v_{m-5,n-1}, \cdots, v_{2,n-1} \in S$.
Thus, m is odd, a contradiction.

Decycling number of Path Product $P_m \square P_n$

New Lower bound

Corollary

For $m \geq 5$*, if* $m \equiv 0$ *(mod* 6*) and* $n \equiv 2$ $($*mod* $3)$ *or* $(m, n) \equiv (3, 2)$ $($*mod* $6)$*, then* $\nabla(P_m \square P_n) \geq F_{m,n} + 1$.

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For $m \geq 5$, if $m \equiv 0$ (mod 6) and $n \equiv 2$ (mod 3) or $(m, n) \equiv (3, 2)$ (mod 6), then $\nabla(P_m \square P_n) \geq F_{m,n} + 1$.

Theorem

For $m \geq 5$, if $(m, n) \equiv (0, 2), (0, 5), (3, 2), (2, 0), (5, 0), (2, 3)$ (mod 6), then $\nabla(P_m \square P_n) = F_{m,n} + 1$.

Decycling number of Path Product $P_m \square P_n$

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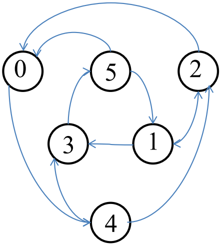
Theorem (Lien, Fu, Shih, 2014, DMAA)

For $m, n \geq 6$, $\nabla(P_m \square P_n) \leq F_{m,n} + 1$.

Decycling number of Digraphs



Decycling number of Digraphs

Notation:

Let v be a vertex in a digraph. The *out-neighborhood* or *successor set* $N^+(v)$ is $\{x \in V(G) : v \to x\}$.

For $S \subseteq V(G)$, $N^+(S) = \cup_{v \in S} N^+(v)$.

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Construction Method

Lemma

Let S be a set of vertices in a digraph G. Then S is a decycling set of G if and only if we can find a sequence of subsets of $V(G)$, $S = S_0, S_1, \cdots, S_t = V(G)$ such that
(1) $S_i \subseteq S_{i+1}$; and
(2) $N^+(S_{i+1} \setminus S_i) \subseteq S_i$ for $i = 0, 1, \cdots, t-1$.

Decycling number of Digraphs

Example 1.

Decycling number of Digraphs

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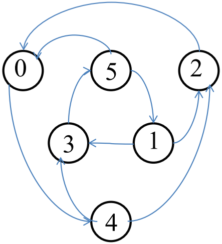

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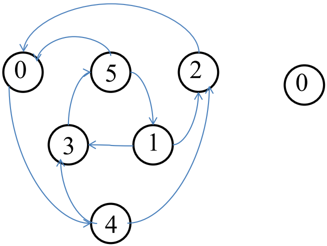

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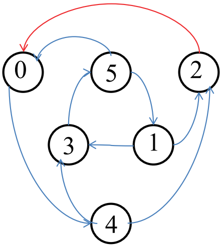

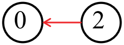

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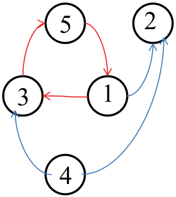
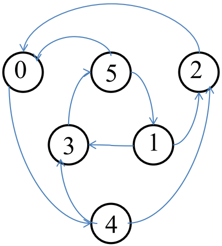


There is a directed cycle $(1, 3, 5)$.

Decycling number of Digraphs

Example 2.



Decycling number of Digraphs

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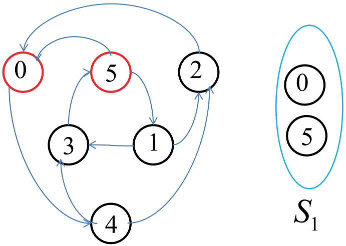

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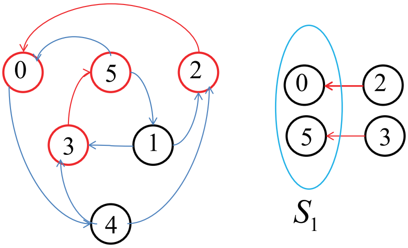

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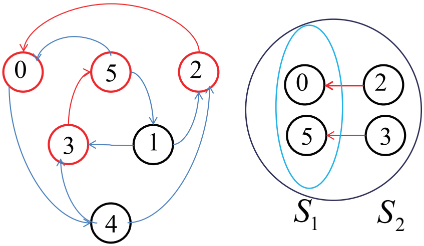

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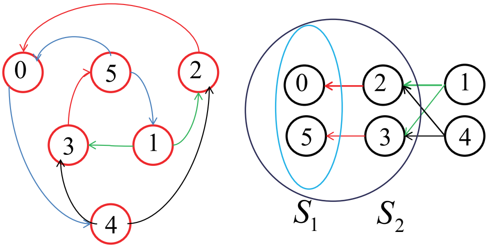

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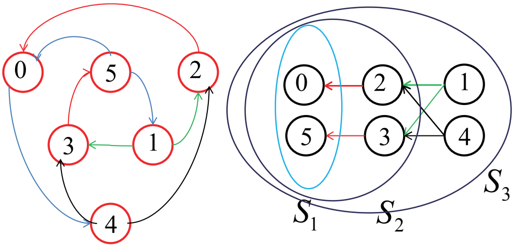

We have $S_3 = V(G)$.

Decycling number of Digraphs

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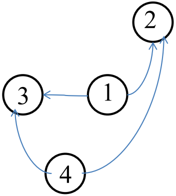

de Bruijn Digraphs and Kautz Digraphs

Definition (de Bruijn digraph $B(d,n)$)

$V(B(d,n)) = \{x_1x_2\cdots x_n : x_i \in \{0,1,\cdots,d-1\}, 1 \le i \le n\}$.
$Edge : X = x_1x_2\cdots x_n \longrightarrow Y = x_2x_3\cdots x_n\alpha$ where $\alpha \in \{0,1,\cdots,d-1\}$.

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Definition (Kautz digraph $K(d, n)$)

$V(K(d, n)) = \{x_1 x_2 \cdots x_n : x_i \in \{0, 1, \cdots, d\}, 1 \leq i \leq n$ *and* $x_i \neq x_{i+1}, 1 \leq i \leq n-1\}$.
$Edge : X = x_1 x_2 \cdots x_n \longrightarrow Y = x_2 x_3 \cdots x_n \alpha$ *where* $\alpha \in \{0, 1, \cdots, d\}$.

Remark: $K(d, n) \subseteq B(d+1, n)$.

Generalized de Bruijn Digraphs Generalized Kautz Digraphs

Definition (Generalized de Bruijn digraph $G_B(d,n)$)

$V(G_B(d,n)) = \{0,1,\cdots,n-1\}$.
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Definition (Generalized Kautz digraph $G_K(d,n)$)

$V(G_K(d,n)) = \{0, 1, \cdots, n-1\}$.
$E(G_K(d,n)) = \{(x,y) | y \equiv -dx - i \pmod{n}, 1 \leq i \leq d\}$.

Decycling number of Generalized Kautz Digraphs

Theorem (Lien, Kuo and Fu)

$$\nabla(G_k(d,n)) \leq \begin{cases} \frac{2}{9}n + 3t + 1, \\ \text{where } n \equiv t \pmod{36}, \text{ for } d = 2, \\ \frac{n}{3} + \frac{9}{4}t + 6 \\ \text{where } n \equiv t \pmod{36}, \text{ for } d = 3, \\ (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d - t + 5) - 2, \\ \text{where } n \equiv t \pmod{d+1}, \text{ for } d \geq 4. \end{cases}$$

Decycling number of Generalized de Bruijn Digraphs

Theorem (Lien, Kuo and Fu)

$\nabla(G_B(d,n)) \leq \frac{d+1}{2d}n + 2(d-1).$

Objective Work

- For a planar graph G, $\nabla(G) \overset{?}{\leq} 2\nu(G)$ (Jones' conjecture 2002).
- For a planar graph G, $\nabla(G) \overset{?}{\leq} |V(G)|/2$ (Albertson and Berman 1979).
- For a bipartite planar graph G, $\nabla(G) \overset{?}{\leq} 3|V(G)|/8$ (Albertson and Berman 1979).
- We have $\lceil \frac{(m-1)(n-1)+1}{3} \rceil \leq \nabla(P_m \Box P_n) \leq \lceil \frac{(m-1)(n-1)+1}{3} \rceil + 1$. Find the exact value of $\nabla(P_m \Box P_n)$.
- Find the lower bound of directed graphs.

References

- Albertson MO, Berman DM (1979) A conjecture on planar graphs, Bondy JA, Murty USR, Graph theory and related topics 357.
- H. Chang, H. L. Fu and M. Y. Lien, The decycling number of outerplanar graphs, J. Comb. Optim. 25 (2013)
- Erdös P, Saks M, Sós VT (1986) Maximum induced trees in graphs. J Combin Theory Ser B 41:61-79.
- Bau S, Beineke LW, Vandell RC (1998) Decycling snakes. Congr Numer 134:79-87.
- Bodlaender HL (1994) On disjoint cycles. Int J Found Comput Sci 5:59-68.

- Erdös P, Saks M, Sós VT (1986) Maximum induced trees in graphs. J Combin Theory Ser B 41:61-79.

- Festa P, Pardalos PM, Resende MGC (2000) Feedback set problems, Handbook of Combinatorial Optimization, Du D-Z, Pardalos PM, Eds, Kluwer Academic Publishers, Supplement A, pp 209-259.
- Kloks T, Lee C-M, Liu J (2002) New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. in Proceedings of the 28th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2002) Springer-Verlag, 2573:282-295.
- M. Y. Lien, H. L. Fu and C. H. Shih, The decycling number of $P_m \square P_n$, Discrete Math., Alg. and Appl. DOI: 10.1142/S1793830914500335, 2014.
- F. R. Madelaine and I. A. Stewart, Improved upper and lower bounds on the feedback vertex numbers of grids and butterflies, Discrete Math., 308 (2008) 4144-4164.

- D. A. Pike and Y. Zou, Decycling Cartesian products of two cycles, SIAM J. Discrete Math., 19 (2005) 651-663.
- X. Xu, Y. Cao, J-M. Xu and Y. Wu, Feedback numbers of de Bruijn digraphs, Computers and Mathematics with Application, 59 (2010) 716-723.
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Thank you for your attention!