A General Framework for Central Limit Theorems of Additive Shape Parameters in Random Digital Trees

(joint work with M. Fuchs)

Chung-Kuei Lee

Department of Applied Mathematics
National Chiao Tung University

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Members in the Digital Tree Family (I)

We are mainly dealing with Tries, PATRICIA Tries, Digital Search Trees (DSTs) and Bucket Digital Search Trees (b-DSTs).

- Tries:
  - Invented by R. de la Briandais and named by E. Fredkin.
  - Have many advantages over other already-existing data structures such as binary search trees.
  - Applied in all kinds of areas in computer science.

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- PATRICIA Tries:
  - Invented by D. R. Morrison in 1986.
  - An variant of Tries which avoid one way branching of internal nodes.
  - Applied in many areas, especially in IP routing.

Members in the Digital Tree Family (II)

- Digital Search Trees:
  - Invented by E. Coffman and J. Eve in 1970.
  - Attracted considerable attention due to their close connection to the famous Lempel-Ziv compression scheme.
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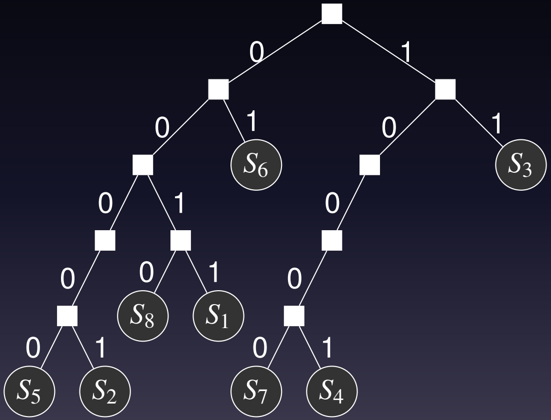
- Bucket Digital Search Trees:
  - A generalization of DSTs.
  - Each node can store b keys. Origin DSTs are b-DSTS with $b = 1$.
  - Related to many practical algorithms, such as memory management in UNIX.

Construction of Tries



$S_1 = 0011010\ldots$
$S_2 = 0000110\ldots$
$S_3 = 1110110\ldots$
$S_4 = 1000100\ldots$
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$S_6 = 0110101\ldots$
$S_7 = 1000011\ldots$
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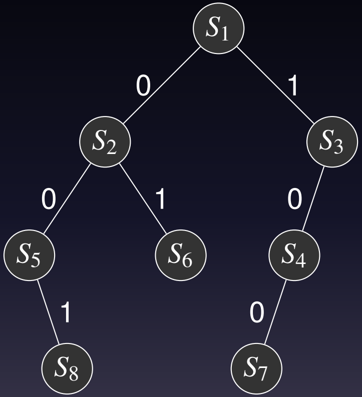
Construction of DSTs



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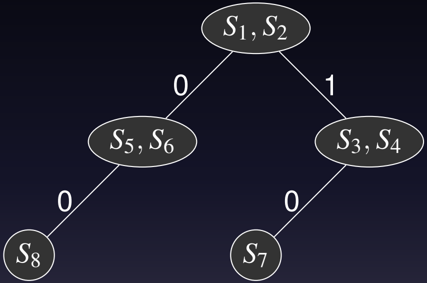
Construction of b-DSTs



$$
\begin{aligned}
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S_8 &= 0010011\ldots
\end{aligned}
$$

Figure: A bucket digital search tree built from the keys $S_1, \ldots, S_8$ with bucket size $b = 2$.

Random Model

Random Model: Bernoulli model.
Bits of keys: i.i.d. Bernoulli random variables

Bianry case: The i-th key will be of the form

$$A_{i,1}, A_{i,2}, \ldots, A_{i,l}, \ldots$$

where $\mathbb{P}(A_{i,j} = 0) = p$ and $\mathbb{P}(A_{i,j} = 1) = q = 1 - p$.

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m-ary case: The i-th key is of the same form with $A_{i,j} \in \mathcal{A} = \{a_1, ..., a_m\}$ for some alphabet $\mathcal{A}$ of the size m.
$\mathbb{P}(A_{i,j} = a_k) = p_k$ with

$$\sum_{i=1}^{m} p_i = 1 \quad \text{and } 0 \leq p_i \leq 1 \text{ for all } i.$$

Additive Shape Parameters

Additive Shape Parameters in Tries A sequence of random variables satisfying the recurrence

$$X_n \stackrel{d}{=} \sum_{r=1}^{m} X^{(r)}_{B_n^{(r)}} + T_n, \quad (n \geq n_0),$$

where $n_0 \geq 0$ is an integer, $X_n, X_n^{(1)}, \ldots, X_n^{(m)}, (B_n^{(1)}, \ldots, B_n^{(m)}), T_n$ are independent and $B_n^{(r)}$ is the multinomial distribution.

Additive Shape Parameters in (Bucket) DSTs A sequence of random variables satisfying the recurrence

$$X_{n+b} \stackrel{d}{=} \sum_{r=1}^{m} X^{(r)}_{B_n^{(r)}} + T_{n+b}, \quad (n \geq n_0),$$

where $b \in \mathbb{N}$.

Contraction Method (I)

We start with a sequence of d-dimensional random vectors $\{Y_n\}_{n\geq 0}$ satisfying the distributional recursion

$$Y_n \stackrel{d}{=} \sum_{r=1}^{k} A_r(n) Y^{(r)}_{I_r^{(n)}} + b_n, \quad n \geq n_0,$$

where

1. $I_r^{(n)}$ is a vector of random cardinalities with $I_r^{(n)} \in \{0, \ldots, n\}$
2. $\left(A_1(n), \ldots, A_k(n), b_n, I_1^{(n)}, \ldots, I_k^{(n)}\right), (Y_n^{(1)}), \ldots, (Y_n^{(k)}), (Y_n)$ are independent,
3. $A_1(n), \ldots, A_k(n)$ are random $d \times d$ matrices,
4. b_n is a random d-dimensional vector,
5. $(Y_n^{(1)}), \ldots, (Y_n^{(k)})$ are identically distributed as (Y_n).

Contraction Method (II)

Next, we normalize the Y_n by

$$X_n := C_n^{-1/2}\left(Y_n - M_n\right), \quad n \geq n_0,$$

where $M_n \in \mathbb{R}^d$ and C_n are suitably chosen positive-definite square matrices. The normalized quantities X_n then satisfy the modified recurrence

$$X_n \stackrel{d}{=} \sum_{r=1}^{k} A_r(n) X_{I_r^{(n)}}^{(r)} + b^{(n)}, \quad n \geq n_0,$$

with

$$A_r^{(n)} := C_n^{-1/2} A_r(n) C_{I_r^{(n)}}^{1/2}, \quad b^{(n)} := C_n^{-1/2}\left(b_n - M_n + \sum_{r=1}^{k} A_r(n) M_{I_r^{(n)}}\right)$$

and the independence relations are as for Y_n. The normalized quantities will converge in ζ_s under suitable conditions.

Contraction Method (III)

Theorem

Let (X_n) be normalized as before and s-integrable and $0 < s \leq 3$. Assume that as $n \to \infty$,

1. $\left(A_1^{(n)}, \dots, A_k^{(n)}, b_n\right) \xrightarrow{\mathcal{L}_s} (A_1^*, \dots, A_k^*, b^*)$,
2. $\mathbb{E} \sum_{r=1}^{k} \|A_r^*\|_{op}^s < 1$, and
3. $\mathbb{E}\left[\mathbf{1}_{\{I_r^{(n)} \leq l\} \cup \{I_r^{(n)} = n\}} \|A_r^{(n)}\|_{op}^s\right] \to 0$ for all $l \in \mathbb{N}$ and $r = 1, \dots, k$.

Then (X_n) converges to a limit X,

$$\zeta_s(X_n, X) \to 0, \quad n \to \infty.$$

Derive the Functional Equations (I)

Consider the moment generating function for X_n:

$$M_n(y) := \mathbb{E}\left(e^{X_n y}\right).$$

The recurrence gives us

$$M_n(y) = \mathbb{E}\left(e^{T_n y}\right) \sum_{j_1+\cdots+j_m=n} \pi_{j_1,\ldots,j_m} M_{j_1}(y) \cdots M_{j_m}(y), \quad (n \geq n_0),$$

where

$$\pi_{j_1,\ldots,j_m} = \binom{n}{j_1 \ldots j_m} p_1^{j_1} \cdots p_m^{j_m}.$$

Now, we can get the recurrences for the first and second moment of X_n by computing $M_n'(0)$ and $M_n''(0)$.

Derive the Functional Equations (II)

We define the poissonized generating functions as the following

$$\tilde{f}_1(z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(X_n) \frac{z^n}{n!}, \qquad \tilde{f}_2(z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(X_n^2) \frac{z^n}{n!}$$

$$\tilde{h}_1(z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(T_n) \frac{z^n}{n!}, \qquad \tilde{h}_2(z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(T_n^2) \frac{z^n}{n!},$$

Then the recurrence relation we get from the moment generating function become

$$\tilde{f}_1(z) = \sum_{r=1}^{m} \tilde{f}_1(p_r z) + \tilde{h}_1(z),$$

$$\tilde{f}_2(z) = \sum_{r=1}^{m} \tilde{f}_2(p_r z) + \sum_{r \neq s} \tilde{f}_1(p_r z) \tilde{f}_1(p_s z) + \tilde{h}_2(z) + \tilde{g}(z).$$

Functional Equations for the Poissonized Variance

Use the idea of Poissonized variance and the functional equations for the first and second moment of X_n, we get:

$$\tilde{V}_X(z) = \sum_{r=1}^{m} \tilde{V}_X(p_r z) + \tilde{V}_T(z) + \tilde{\phi}_1(z) + \tilde{\phi}_2(z),$$

where

$$\begin{aligned}
\tilde{\phi}_1(z) =& \tilde{g}(z) - 2\tilde{h}_1(z) \sum_{r=1}^{m} \tilde{f}_1(p_r z) - 2z\tilde{h}_1'(z) \sum_{r=1}^{m} p_r \tilde{f}_1'(p_r z), \\
\tilde{\phi}_2(z) =& z \sum_{r<s} p_r p_s \left(\tilde{f}_1'(p_r z) - \tilde{f}_1'(p_s z) \right)^2 .
\end{aligned}$$

JS-admissibility

A systematic method which helps researchers using analytical-depoissonization.

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Definition

We let $\epsilon, \epsilon' \in (0,1)$ be arbitrarily small numbers. An entire function $\tilde{f}$ is said to be JS-admissible, denoted by $\tilde{f} \in \mathscr{J}\mathscr{S}_{\alpha,\beta}$, if the following two conditions hold for $|z| \geq 1$.

(I) There exists $\alpha, \beta \in \mathbb{R}$ such that uniformly for $|\arg(z)| \leq \epsilon$,

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha}(\log_{+}|z|)^{\beta}\right),$$

where $\log_{+} x := \log(1+x)$.

(O) Uniformly for $\epsilon \leq |\arg(z)| \leq \pi$,

$$f(z) := e^{z}\tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon')|z|}\right).$$

Lemma for the Functional Equations

Let $\tilde{f}(z)$ and $\tilde{h}(z)$ be entire functions satisfying a functional equation of the form

$$\tilde{f}(z) = \sum_{r=1}^{m} \tilde{f}(p_r z) + \tilde{h}(z)$$

where $h = -\sum_{r=1}^{m} p_r \log p_r$. If $\tilde{h}(z) \in \mathscr{J}\mathscr{S}_{\alpha,\gamma}$ with $0 \leq \alpha < 1$ and $\tilde{f}(0) = \tilde{f}'(0) = 0$, then

$$\tilde{f}(z) = \frac{1}{h} \sum_{\omega_k \in \mathcal{Z}_{<-\alpha-\epsilon}} G(\omega_k) z^{-\omega_k} + \mathcal{O}(z^{\alpha+\epsilon}),$$

where the sum expression is infinitely differentiable and

$$G(\omega) = \int_0^\infty z^{\omega-1} \tilde{h}(z) dz = \mathscr{M}[\tilde{h}; \omega].$$

Asymptotic for Mean and Variance

If $\tilde{h}_1(z) \in \mathscr{J}\mathscr{S}_{\alpha_1,\gamma_1}$ with $0 \leq \alpha_1 < 1$, then

$$\mathbb{E}\left(X_n\right) = \frac{1}{h} \sum_{\omega_k \in \mathcal{Z}_{<-\alpha_1-\epsilon}} G_E(\omega_k) n^{-\omega_k} + \mathcal{O}(n^{\alpha_1+\epsilon}),$$

where the sum expression is infinitely differentiable and

$$G_E(\omega) = \mathscr{M}[\tilde{h}_1;\omega] = \int_0^\infty \tilde{h}(z) z^{\omega-1} dz.$$

Moreover, if $\tilde{V}_T(z) \in \mathscr{J}\mathscr{S}_{\alpha_2,\gamma_2}$ with $0 \leq \alpha_2 < 1$ and $\tilde{h}_2(z) \in \mathscr{J}\mathscr{S}$, then

$$\operatorname{Var}\left(X_n\right) \sim \frac{1}{h} \sum_{\omega_k \in \mathcal{Z}_{=-1}} G_V(\omega_k) n^{-\omega_k},$$

where the sum expression is infinitely differentiable and

$$G_V(\omega) = \mathscr{M}[\tilde{V}_T + \tilde{\phi}_1 + \tilde{\phi}_2;\omega] = \int_0^\infty \left(\tilde{V}_T(z) + \tilde{\phi}_1(z) + \tilde{\phi}_2(z)\right) z^{\omega-1} dz.$$

General Central Limit Theorem for Tries

Theorem

Suppose that $\tilde{h}_1(z) \in \mathscr{J}\mathscr{S}_{\alpha_1,\gamma_1}$ with $0 \leq \alpha_1 < 1/2$, $\tilde{h}_2(z) \in \mathscr{J}\mathscr{S}$ and $\tilde{V}_T(z) \in \mathscr{J}\mathscr{S}_{\alpha_2,\gamma_2}$ with $0 \leq \alpha_2 < 1$. Moreover, we assume that $\|T_n\|_s = o(\sqrt{n})$ with $2 < s \leq 3$ and $\mathbb{V}(X_n) \geq cn$ for all n large enough and some $c > 0$. Then, as $n \to \infty$,

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Differential Functional Equations for DSTs

By similar computations, we get that

$$\tilde{f}_1(z) + \tilde{f}_1'(z) = 2\tilde{f}_1\left(\frac{z}{2}\right) + \tilde{\tau}_1(z),$$

and

$$\begin{aligned}\tilde{V}(z) + \tilde{V}'(z) =& 2\tilde{V}\left(\frac{z}{2}\right) + \tilde{V}_T(z) + \tilde{\lambda}(z) - 4\tilde{\tau}_1(z)\tilde{f}_1\left(\frac{z}{2}\right) \\ & - 2z\tilde{\tau}_1'(z)\tilde{f}_1'\left(\frac{z}{2}\right) + z\tilde{f}_1''(z)^2\end{aligned}$$

where $\tilde{V}_T(z) = \tilde{\tau}_2(z) - \tilde{\tau}_1(z)^2 - z\tilde{\tau}_1'(z)^2$.

General Central Limit Theorems for DSTs

Theorem

Suppose $\tilde{\tau}_1(z) \in \mathscr{J}\mathscr{S}_{\alpha_1,\gamma_1}$ with $0 \le \alpha_1 < 1/2$, $\tilde{\tau}_2(z) \in \mathscr{J}\mathscr{S}$ and $\tilde{V}_T(z) \in \mathscr{J}\mathscr{S}_{\alpha_2,\gamma_2}$ with $0 \le \alpha_2 < 1$. Moreover, we assume that $\|T_n\|_s = o(\sqrt{n})$ with $2 < s \le 3$ and $\mathbb{V}(X_n) \ge cn$ for all n large enough and some $c > 0$. Then, as $n \to \infty$

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Lower Bound for Variance

Consider two nonnegative sequence $\{\alpha_i\}$ and $\{\beta_i\}$ satisfying a recurrence of the form

$$\alpha_{n+1} = \sum_{i=1}^{m} a_i \sum_{j=0}^{n} f(n, j, p_i)\alpha_j + \beta_n, \quad (n \geq n_0),$$

where $a_1, \ldots, a_m$ are positive real numbers, $p_i \in (0, 1)$ for all $1 \leq i \leq m$ and $f(n, j, p)$ is a nonnegative-valued function. We assume that there exists some $j' \geq n_0$ such that $\beta_{j'} > 0$. We also assume that $f(n, j, p)$ satisfies that $\sum_{j=0}^{n} f(n, j, p) = 1$ and there exists $n_1 \geq n_0$ such that for all $n > n_1$ and $p < 1$,

$$\sum_{|j-pn|>pn^{\tau}} f(n, j, p) = \mathcal{O}(n^{\tau-1})$$

for some constant $1 > \tau > 0$, then $\alpha_n = \Omega(n^{\lambda})$ with λ being the unique real root of $F(z) = 1 - \sum_{i=1}^{m} a_i p_i^z$.

The Internal Nodes of Tries

$N_n^{(k)}$: The number of internal nodes of outdegree k in a trie built on n keys.

We will give a multivariate study of these parameters by considering

$$Z_n = \sum_{k=1}^{m} a_k N_n^{(k)},$$

where $a_1, \ldots, a_m$ are arbitrary real number with $a_i \neq (i-1)a_2$ for some i.

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Lemma

Z_n is not deterministic for n large enough.

proof

Find two examples.

Recurrence of Z_n

We derive the recurrence for Z_n from $N_n^{(k)}$:

$$N_n^{(k)} \stackrel{d}{=} \sum_{i=1}^{m} (N_{I_n^{(i)}}^{(k)})^{(i)} + T_n^{(k)}, \qquad (n \geq 2),$$

with the initial conditions $N_0^{(k)} = N_1^{(k)} = 0$ and

$$T_n^{(k)} = \begin{cases} 1, & \text{if } \#\{1 \leq i \leq m \ : \ I_n^{(i)} \neq 0\} = k; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$Z_n \stackrel{d}{=} \sum_{i=1}^{m} Z_{I_n^{(i)}}^{(i)} + T_n, \qquad (n \geq 2),$$

where $Z_0 = Z_1 = 0$ and

$$T_n = \sum_{k=1}^{m} a_k T_n^{(k)}.$$

The Lower Bound of the Variance of Z_n

Theorem

We have, $\mathrm{Var}(Z_n) \geq cn$ with $c > 0$ for all n large enough.

proof

- Set $\mu_n = \mathbb{E}(Z_n)$ and

$$M_n(y) = \mathbb{E}\left(e^{(Z_n-\mu_n)y}\right).$$

- From the recurrence, we get

$$\mathrm{Var}(Z_n) = \sigma_n^2 = \sum_{i=1}^{m}\sum_{j=0}^{n}\binom{n}{j}p_i^j(1-p_i)^{n-j}\sigma_j^2 + \eta_n, \qquad (n \geq 2)$$

- Check that $\eta_n > 0$ for some $n > 2$, then apply the lemma.

Main Theorem from the Framework

Theorem

We have, as $n \to \infty$,

$$\mathbb{E}(Z_n) \sim nP(\log_{1/a} n), \qquad \mathrm{Var}(Z_n) \sim nQ(\log_{1/a} n),$$

where $a > 0$ is a suitable constant and $P(z), Q(z)$ are infinitely differentiable, 1-periodic functions (possibly constant). Moreover, $\mathrm{Var}(Z_n) > 0$ for all n large enough and

$$\frac{Z_n - \mathbb{E}(Z_n)}{\sqrt{\mathrm{Var}(Z_n)}} \xrightarrow{d} N(0, 1).$$

Proof

Check all the conditions in the framework.

Bivariate Setting

Set

$$\mathrm{Var}(N_n^{(k_1)}) \sim nQ^{(k_1)}(\log_{1/a} n), \quad \mathrm{Var}(N_n^{(k_2)}) \sim nQ^{(k_2)}(\log_{1/a} n)$$

and

$$\Sigma_n = \begin{pmatrix} nQ^{(k_1)}(\log_{1/a} n) & nQ^{(k_1,k_2)}(\log_{1/a} n) \\ nQ^{(k_1,k_2)}(\log_{1/a} n) & nQ^{(k_2)}(\log_{1/a} n) \end{pmatrix}.$$

Lemma

The correlation coefficient $\rho(N_n^{(k_1)}, N_n^{(k_2)})$ is not -1 or 1 for all n large enough.

proof

Check examples.

Bivariate Limit Law

Theorem

Assume that $(k_1, k_2, m) \notin \{(1,2,2),(2,3,3)\}$. Then, Σ_n is positive definite for n large enough and, as $n \to \infty$,

$$\Sigma_n^{-1/2} \begin{pmatrix} N_n^{(k_1)} - \mathbb{E}(N_n^{(k_1)}) \\ N_n^{(k_2)} - \mathbb{E}(N_n^{(k_2)}) \end{pmatrix} \xrightarrow{d} N(0, I_2),$$

where I_2 denotes the 2×2 identity matrix.

Proof

By checking the required conditions for the contraction method.

2-Protected Nodes in DSTs

k-protected node: A node in a tree is said to be k-protected if the distance from the node to each leaf is at least k.

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- Proposed by Cheon and Shapiro.
- Studied under Planar trees, Motzkin trees, ternary trees, k-ary trees (mean), binary search trees (CLT), k-ary search trees (some probabilistic properties), DSTs (mean).
- Janson and Devroye extend the concept to **protected fringe subtree**.

Distribution Recurrence of 2-protected Nodes in DSTs

Let L_n be the number of 2-protected nodes in a DST built on n keys, then under the Bernoulli model, 2-protected nodes in DSTs satisfies the recurrence

$$L_{n+1} \stackrel{d}{=} L_{B_n} + L^*_{n-B_n} + T_n, \quad (n \geq 3),$$

where $B_n = \text{Binomial}(n, 1/2)$ and

$$T_n = \begin{cases} 0, & B_n = 1 \vee n-1; \\ 1, & \text{Otherwise.} \end{cases}$$

$\Longrightarrow$ The framework can be applied here.

Results from the Framework

Theorem

We have that

$$\mathbb{E}(L_n) = \quad \frac{n}{\log 2} \sum_{k \in \mathbb{Z}} \frac{G_E(2+\chi_k)}{\Gamma(2+\chi_k)} n^{\chi_k} + \mathcal{O}(n^{\epsilon}) \quad \text{as } n \to \infty,$$

and

$$\mathbb{V}(L_n) \sim \quad \frac{n}{\log 2} \sum_{k \in \mathbb{Z}} \frac{G_V(2+\chi_k)}{\Gamma(2+\chi_k)} n^{\chi_k} \qquad \text{as } n \to \infty.$$

Moreover,

$$\frac{L_n - \mathbb{E}(L_n)}{\sqrt{\mathrm{Var}(L_n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Expression of $G_E(\omega)$

We have that

$$G_E(\omega) = \kappa(-\omega)\Gamma(\omega)\Gamma(1-\omega) + \frac{Q(2^{\omega-1})}{Q(1)}\Gamma(-\omega)\Gamma(\omega+1),$$

where

$$\begin{aligned}\kappa(\omega) =& \frac{8\cdot 2^{4l} - 32\cdot 2^{3l} + 46\cdot 2^{2l} - 32\cdot 2^{l} + 9}{2^{1-l\omega}(2\cdot 2^{l}-1)^2(2^{l}-1)^3} \\ &- \frac{2^{\omega+l+3}\left(\omega(2^{l+1}-1) + 2^{l+1} - 2\right)}{(2\cdot 2^{l}-1)^2} \\ &+ \frac{2^{l}\left(2^{l}(\omega^2+3\omega-2) - 2^{l+1}(\omega^2+4\omega-2) + \omega^2+5\omega+2\right)}{4(2^{l}-1)^3}.\end{aligned}$$