Which connected graphs are determined by their distance spectra

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Joint work with Jinlong Shu, Yuan Hong, Jianfeng Wang, Shicai Gong, Mingqing Zhai

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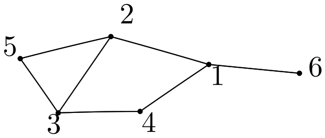
Definition

The distance matrix $D(G) = (d_{ij})_{n\times n}$ of a connected graph G is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j.

Since $D(G)$ is irreducible nonnegative matrix, we call order the distance spectra (D-eigenvalue) of a connected graph G as: $\lambda_1(D) > \lambda_2(D) \geq \cdots \geq \lambda_{n-1}(D) \geq \lambda_n(D)$.



$$D(G) = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 1 & 2 & 0 & 3 \\ 1 & 2 & 2 & 2 & 3 & 0 \end{pmatrix}$$

The research for distance matrix can be dated back to the following papers, which present an interesting result that the determinant of the distance matrix of trees with order n is always $(-1)^{n-1}(n-1)2^{n-2}$, independent of the structure of the tree.

[M. Edelberg, M.R. Garey, R.L. Graham, On the distance matrix of a tree, Discrete Math. 14 (1976) 23-29.]

[R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell Syst. Tech. J. 50 (1971) 2495-2519.]

Two nonisomorphic graphs with the same D-spectra are called D-cospectral.

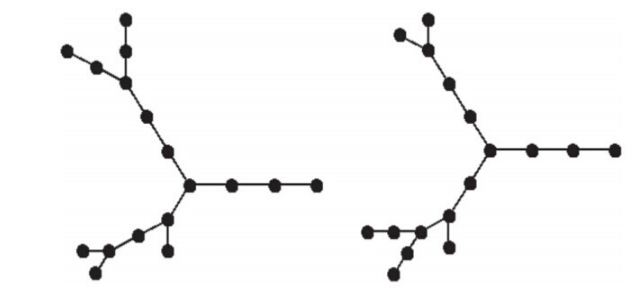


Fig. 1 The smallest D-cospectral trees.

[M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra and its Applications 439 (2013) 21-33.]

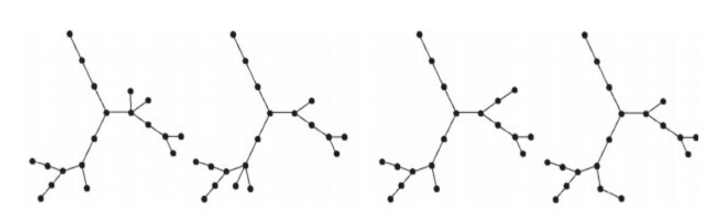


Fig. 2 Two pairs of D-cospectral trees on 19 vertices.

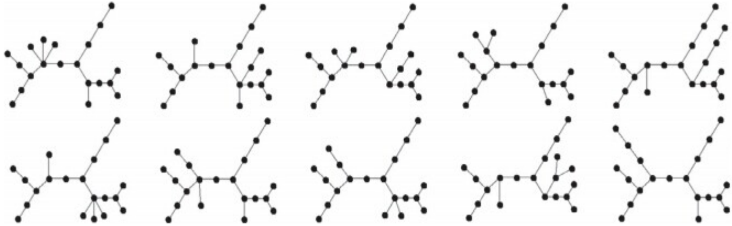


Fig. 3 Five pairs, by column, of D-cospectral trees on 20 vertices.

We say that a graph is determined by the D-spectra if there is no other nonisomorphic graph with the same D-spectra.

Problem

Which connected graphs are determined by their D-spectra?

The complete graph K_n (is the unique graph which the least D-eigenvalue attains the maximum among all connected graphs.) and the path P_n (is the unique graph which the largest D-eigenvalue attains the maximum among all connected graphs.) are determined by their D-spectra.

Theorem

Let G be a connected graph and D be the distance matrix of G. Then $\lambda_n(D) = -2$ with multiplicity $n - k$ if and only if G is a complete k-partite graph for $2 \leq k \leq n - 1$.

[H. Lin, Y. Hong, J. Wang, J. Shu, On the distance spectrum of graphs, Linear Algebra and its Applications, 439(2013) 1662-1669]

Cauchy Interlace Theorem

Let A be a Hermitian matrix of order n and let B be a principal submatrix of A of order m. If $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ lists the eigenvalues of A and $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$ the eigenvalues of B, then $\lambda_{n-m+i}(A) \leq \mu_i(B) \leq \lambda_i(A)$ for $i = 1, \cdots, m$.

Proposition

Let G be a connected graph with diameter $d \geq 3$. Then

$$\lambda_n(D(G)) \leq \lambda_4(D(P_4)) = -2 - \sqrt{2}.$$

Conjecture

For a connected graph G, $d \leq -\lambda_n(D)$, where d denotes the diameter of G. Equality holds if and only if G is a multipartite graph.

[M. Aouchiche, P. Hansen, Distance spectra of graphs: A survey, Linear Algebra Appl. 458 (2014) 301-386.]

Conjecture

Let G be a connected graph on $n \geq 3$ vertices with girth $g \geq 5$ and minimum dual degree $\delta^\star$. Then $\lambda_n(D) \leq \delta^\star$ where $\delta^\star$ denotes the minimum average 2-degree.

[S. Fajtlowicz, Written on the wall: conjectures derived on the basis of the program Galatea Gabriella Graffiti, technical report, University of Houston, 1998.]

Courant-Weyl inequalities

Let A and B be $n \times n$ Hermitian matrices and $C = A + B$. Then

$$\lambda_i(C) \leq \lambda_j(A) + \lambda_{i-j+1}(B) \ (n \geq i \geq j \geq 1),$$

$$\lambda_i(C) \geq \lambda_j(A) + \lambda_{i-j+n}(B) \ (1 \leq i \leq j \leq n).$$

In either of these inequalities equality holds if and only if there exists a nonzero n-vector that is an eigenvector to each of the three involved eigenvalues.

[W. So, *Commutativity and spectra of Hermitian matrices*, Linear Algebra Appl. 212-213 (1994) 121-129.]

Proposition

Let G be a connected graph with diameter $d = 2$. Then

$$D(G) = J - I + A(\overline{G}).$$

If $G = K_{n_1,\cdots,n_k}$ with $n_1 \geq \cdots \geq n_k$, then $\overline{G} = K_{n_1} \cup \cdots \cup K_{n_k}$. Obviously, $Sp(\overline{G}) = \{n_1 - 1, \cdots, n_k - 1, -1, \cdots, -1\}$.

$\lambda_{i-1}(\overline{G}) + \lambda_2(J - I) \geq \lambda_i(D) \geq \lambda_i(\overline{G}) + \lambda_n(J - I)$ for $i = 2, \cdots, n$.

Conjecture

The complete multipartite graph $K_{n_1,\cdots,n_k}$ is determined by its distance spectra.

In 2014, Y.-L. Jin and X.-D. Zhang confirmed the conjecture.

[Y.-L. Jin, X.-D. Zhang, Complete multipartite graphs are determined by their distance spectra, Linear Algebra Appl. 448 (2014) 285-291.]

Theorem, H. Lin, submitted

Let G be a connected graph and D be the distance matrix of G. Then $\lambda_n(D) \in [-1-\sqrt{2}, -2.383]$ if and only if $G = G(s, n_1, \cdots, n_k)$ for $s \geq 1$. Moreover, $\lambda_n(D) = -1-\sqrt{2}$ with multiplicity $s-1$.

Let $G(s, n_1, \cdots, n_k)$ be the graph obtained by $\underbrace{(K_1 \cup K_2) \vee \cdots \vee (K_1 \cup K_2)}_{s} \vee K_{n_1,\cdots,n_k}$. The following picture shows the graphs with the least D-eigenvalue in each interval.

$\lambda_n(D)$ — $-1-\sqrt{2}$ — -2.383 — -2 — -1

$G(s, n_1, \cdots, n_k), s \geq 2$ | $G(1, n_1, \cdots, n_k)$ | $G(1,1)$ | $K_{n_1,\cdots,n_k}$ | K_n

Theorem, submitted

Let G be a connected graph D be the distance matrix of G. If $\lambda_n(D) \geq -1-\sqrt{2}$, then G is determined by its distance spectra.

For a connected graph G. Note that $\lambda_n(D) \leq -1$.

Problem

For a given k and a sufficient larger n (with respect to k), does $\lambda_{n-k}(D) \leq -1$.

We give a positive answer to the problem for $k = 1, 2$.

Let $K_{s,t}^{r} = K_r \vee (K_s \cup K_t)$ with $r \geq 1$.

Theorem

Let G be a connected graph with order n and D be the distance matrix of G. If $n \geq 4$, then $\lambda_{n-1}(D) \leq -1$ and the equality holds if and only if $G \cong K_{s,t}^{r}$ with $r \geq 1$.

Lemma

Let G be a connected graph with order $n \geq 3$. Then $G \cong K_{s,t}^{r}$ if and only if G is $\{K_{1,3}, P_4, C_4\}$-free.

[H. Lin, M. Zhai, S. Gong, On graphs with at least three distance eigenvalues less than -1, Linear Algebra and its Applications, 458 (2014) 548-558.]

Theorem

Let G be a connected graph with order n and D be the distance matrix of G. If $n \geq 7$, then $\lambda_{n-2}(D) \leq -1$.

Theorem

For any non-negative integers r, s, t with $r \geq 1$, the graph $K_{s,t}^{r}$ is determined by its distance spectra.

Theorem

Let G be a connected graph with order $n \geq 4$ and $\lambda_{n-2}(D(G)) > -1$. Then G is determined by its distance spectra.

Thank You !