Spectral Excess Theorem and its Applications

李光祥 (交大應數)

2014 組合數學新苗研討會
台灣師範大學

August 3, 2014

This research is conducted under the supervision of
Professor Chih-wen Weng.

Distance-regularity

- Let G be a connected graph on n vertices, with vertex set V and diameter D.
- For $0 \leq i \leq D$ and two vertices $u, v \in V$ at distance i, set

$$c_i(u,v) := |G_1(v) \cap G_{i-1}(u)|,$$
$$a_i(u,v) := |G_1(v) \cap G_i(u)|, \text{ and}$$
$$b_i(u,v) := |G_1(v) \cap G_{i+1}(u)|.$$

- These parameters are well-defined if they are independent of the choice of u, v. In this case we use the symbols c_i, a_i and b_i for short.
- A connected graph G with diameter D is called distance-regular if the above-mentioned parameters are well-defined.

- Assume that adjacency matrix A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $m_0 = 1$, m_1, …, m_d.

- The spectrum of G is denoted by the multiset

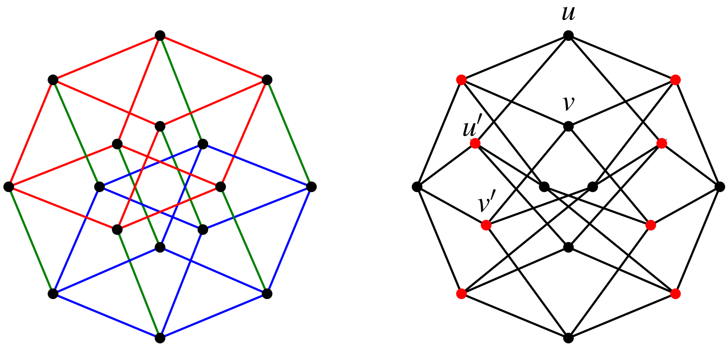
$$\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}.$$

- The parameter d is called the spectral diameter of G.

  Note that $D \leq d$.

Question: Is the distance-regularity of a graph determined by its spectrum?

Answer: In general, the answer is negative.



The Hamming 4-cube and the Hoffman graph
(distance-regular) (c_2 is not well-defined)

We have known that the distance-regularity of a graph is in general not determined by its spectrum.

Question: Under what additional conditions, the answer is positive?

Answer: The spectral excess theorem.

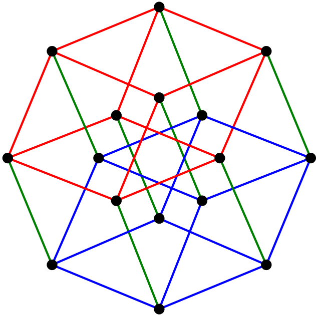
The spectral excess theorem gives a quasi-spectral characterization for a regular graph to be distance-regular.

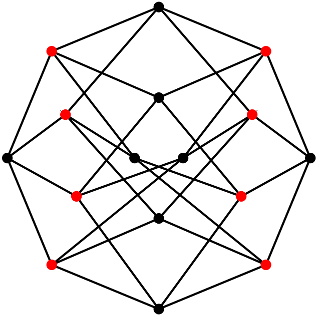
Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. □

- $\overline{k}_d = \frac{1}{n}\sum_{u \in V} |G_d(u)|$: average excess (combinatorial aspect) – the mean of the numbers of vertices at distance d from each vertex
- $p_d(\lambda_0)$: spectral excess (algebraic aspect) – a number which can be computed from the spectrum

Therefore, besides the spectrum, a simple combinatorial property suffices for a regular graph to be distance-regular.





The Hamming 4-cube and the Hoffman graph

$(\bar{k}_d = 1 = p_d(\lambda_0))$ $(\bar{k}_d = 1/2 < 1 = p_d(\lambda_0))$

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. □

The following example shows that the spectral excess theorem cannot be directly applied to nonregular graphs.

Example

Let G be a path on three vertices, with spectrum $\{\sqrt{2}, 0, -\sqrt{2}\}$. Note that $D = d = 2$, $\overline{k}_2 = 2/3$ and $p_2(\lambda_0) = 1/2$. This shows that $\overline{k}_d \leq p_d(\lambda_0)$ does not hold for nonregular graphs.

Thus, a 'weighted' version of the spectral excess theorem is given in order to make it applicable to nonregular graphs.

Two kinds of inner products

- Consider the vector space $\mathbb{R}_d[x]$ consisting of all real polynomials of degree at most d with the inner product

$$\langle p, q\rangle_G := \operatorname{tr}(p(A)q(A))/n = \sum_{u,v}(p(A) \circ q(A))_{uv}/n,$$

  for $p, q \in \mathbb{R}_d[x]$, where $\circ$ is the entrywise product of matrices.

- For any two $n \times n$ symmetric matrices M, N over $\mathbb{R}$, define the inner product

$$\langle M, N\rangle := \frac{1}{n}\sum_{i,j}(M \circ N)_{ij},$$

  where " $\circ$ " is the entrywise product of matrices.

- Thus $\langle p, q\rangle_G = \langle p(A), q(A)\rangle$ for $p, q \in \mathbb{R}_d[x]$.

Predistance polynomials

By the Gram–Schmidt procedure, there exist polynomials $p_0, p_1, \dots, p_d$ in $\mathbb{R}_d[x]$ satisfying

$$\deg p_i = i \qquad \text{and} \qquad \langle p_i, p_j \rangle_G = \delta_{ij} p_i(\lambda_0)$$

for $0 \le i, j \le d$, where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

These polynomials are called the predistance polynomials of G.

Lemma (Fiol and Garriga, 1997)

The spectral excess $p_d(\lambda_0)$ can be expressed in terms of the spectrum, which is

$$p_d(\lambda_0) = \frac{n}{\pi_0^2} \left(\sum_{i=0}^{d} \frac{1}{m_i \pi_i^2} \right)^{-1},$$

where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$ for $0 \le i \le d$. □

Preparation for the 'weighted' version

- Let α be the eigenvector of A (usually called the Perron vector) corresponding to λ_0 such that $\alpha^t\alpha = n$ and all entries of α are positive. Note that $\alpha = (1,1,\ldots,1)^t$ iff G is regular.

- For a vertex u, let α_u be the entry corresponding to u in α.

  The matrix $\widetilde{A}_i$ with rows and columns indexed by the vertex set V such that

  $$(\widetilde{A}_i)_{uv} = \begin{cases} \alpha_u\alpha_v & \text{if } \partial(u,v) = i, \\ 0 & \text{otherwise} \end{cases}$$

  is called the weighted distance-i matrix of G.

- If G is regular, then $\widetilde{A}_i$ is binary and hence the distance-i matrix A_i.

  Thus, $\widetilde{A}_i$ can be regarded as a 'weighted' version of A_i.

Spectral excess theorem and its ‘weighted’ version

Define $\widetilde{\delta}_i := \langle \widetilde{A}_i, \widetilde{A}_i \rangle$. If G is regular, then

$$\widetilde{\delta}_d = \langle \widetilde{A}_d, \widetilde{A}_d \rangle = \langle A_d, A_d \rangle = \frac{1}{n} \sum_{u \in VG} |G_d(u)| = \overline{k}_d.$$

Hence, $\widetilde{\delta}_d$ can be viewed as a generalization of average excess $\overline{k}_d$.

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. $\square$

Weighted spectral excess theorem (Lee and Weng, 2012)

Let G be a graph with $d+1$ distinct eigenvalues. Then, $\widetilde{\delta}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. $\square$

An application: graphs with odd-girth $2d+1$

Odd-girth theorem (van Dam and Haemers, 2011)

A connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular. $\square$

- In the same paper, they posed the question to determine whether the regularity assumption can be removed.
- Moreover, they showed that the answer is affirmative for the case $d+1=3$, and claimed to have proofs for the cases $d+1 \in \{4,5\}$.
- As an application of the 'weighted' spectral excess theorem, we show that the regularity assumption is indeed not necessary.

Odd-girth theorem (Lee and Weng, 2012)

A connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular. $\square$

We then apply this line of study to the class of bipartite graphs.

- The distance-2 graph G^2 of G is the graph whose vertex set is the same as of G, and two vertices are adjacent in G^2 if they are of distance 2 in G.
- For a connected bipartite graph, the halved graphs are the two connected components of its distance-2 graph.
- For an integer $h \leq d$, we say that G is weighted h-punctually distance-regular if $\widetilde{A}_h = p_h(A)$.

Proposition (BCN, Proposition 4.2.2, p.141)

Suppose that G is a connected bipartite graph. If G is distance-regular, then its two halved graphs are distance-regular. $\square$

Problem (The converse statement)

Suppose that G is a connected bipartite graph, and both halved graphs are distance-regular.

- Can we say that G is distance-regular?
- If not, what additional conditions do we need?

Answer:

- Three examples will be given to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular.

- We will give a quasi-spectral characterization of a connected bipartite weighted 2-punctually distance-regular graph whose halved graphs are distance-regular.

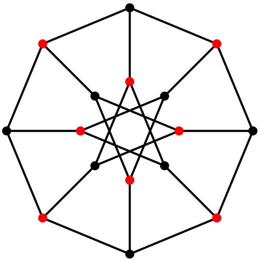
- In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

Three counterexamples

Example 1 (weighted 2-punctually distance-regular & odd spectral diameter)

The Möbius–Kantor graph, with spectrum $\{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$.

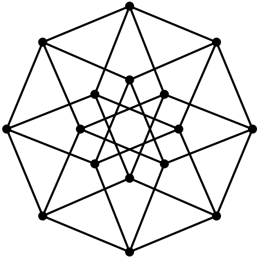
- $D = 4 < 5 = d$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$, and
- both halved graphs the complete multipartite graphs $K_{2,2,2,2}$ (with spectrum $\{6^1, 0^4, (-2)^3\}$), which are distance-regular.

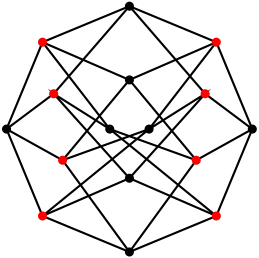


Example 2 (not weighted 2-punctually distance-regular & even spectral diameter)

Consider the Hoffman graph with spectrum $\{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, which is cospectral to the Hamming 4-cube but not distance-regular.

- $D = d = 4$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 3\}$ ($i \neq 2$), and
- its two halved graphs are the complete graph K_8 and the complete multipartite graph $K_{2,2,2,2}$, which are both distance-regular.

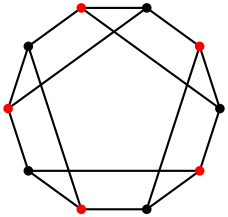




Example 3 (not weighted 2-punctually distance-regular & odd spectral diameter)

Consider the graph obtained by deleting a 10-cycle from the complete bipartite graph $K_{5,5}$, with spectrum $\{3^1, ((\sqrt{5}+1)/2)^2, ((\sqrt{5}-1)/2)^2, ((-\sqrt{5}+1)/2)^2, ((-\sqrt{5}-1)/2)^2, (-3)^1\}$.

- $D = 3 < 5 = d$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1\}$ ($i \neq 2$), and
- both halves graphs are the complete graphs K_5, which are distance-regular.



The remaining case

We have considered three counterexamples.

Example 1 (weighted 2-punctually distance-regular & odd spectral diameter)
Example 2 (not weighted 2-punctually distance-regular & even spectral diameter)
Example 3 (not weighted 2-punctually distance-regular & odd spectral diameter)

Note that the remaining case is that

G is weighted 2-punctually distance-regular with even spectral diameter.

Question: How about the remaining case?

Answer: Under these additional conditions, the converse statement is true.

Theorem (Lee and Weng, 2014)

Suppose that G is a connected bipartite graph, and both halved graphs are distance-regular. If G is weighted 2-punctually distance-regular with even spectral diameter, then G is distance-regular.

Sketch of proof

- By the weighted 2-punctually distance-regularity assumption,
  - G is regular, and
  - both halved graphs have the same spectrum, and thus have the same (pre)distance-polynomials.
- By the above results and the even spectral diameter assumption,
  - $\widetilde{\delta}_d = p_d(\lambda_0)$, and the result follows by (weighted) spectral excess theorem. □

Thank you for your listening!