Covering Problems in Graphs

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Outline

Vertex-version covering

- Let $G = (V, E)$ be a graph.
- Let $\mathcal{P}$ be a property of the vertex set $V(G)$.
- A **vertex-version $\mathcal{P}$-covering** (or simply, **$\mathcal{P}$-covering**) of a graph G is a set $L = \{V_1, \dots, V_s\}$ of vertex subsets of $V(G)$, where each vertex subset satisfies the property $\mathcal{P}$, and

$$\bigcup_{i=1}^{s} V_i = V(G).$$

Vertex-version covering

- Let $\mathcal{C}(\mathcal{P}, G)$ be the set of all $\mathcal{P}$-coverings of G.
- Usually the goal is finding the $\mathcal{P}$-**covering number**, that is,

$$\min_{L=\{V_1,\ldots,V_s\}\in\mathcal{C}(\mathcal{P},G)} f(V_1,\ldots,V_s)$$

for some function f.

Edge-version covering

- Similarly, let $\mathcal{P}'$ be a property of the edge set $E(G)$.
- A **edge-version** $\mathcal{P}'$**-covering** (or simply, $\mathcal{P}'$**-covering**) of a graph $G = (V, E)$ is a set $L' = \{E_1, \ldots, E_t\}$ of edge subsets of $E(G)$, where each edge subset satisfies the property $\mathcal{P}'$, and

$$\bigcup_{i=1}^{t} E_i = E(G).$$

Edge-version covering

- Let $\mathcal{C}'(\mathcal{P}', G)$ be the set of all $\mathcal{P}'$-coverings of G.
- Usually the goal is finding the $\mathcal{P}'$-**covering number**, that is,

$$\min_{L'=\{E_1,\dots,E_t\}\in\mathcal{C}'(\mathcal{P}',G)} f(E_1,\dots,E_t)$$

for some function f.

Examples - edge cover

- An **edge cover** of a graph G is a set of edges $\mathcal{C}$ such that each vertex in G is incident with at least one edge in $\mathcal{C}$, that is,

$$\bigcup_{e \in \mathcal{C}} N(e) = V(G)$$

  where $N(e)$ be the set of vertices which is incident to e.
- A **minimum edge covering** is an edge covering of smallest possible size.
- The **edge covering number** $\beta'(G) = \min_{\mathcal{C}} |C|$ is the size of a minimum edge covering.

Examples - clique cover

- A **clique cover** of a graph G is a set of cliques $\mathcal{C}$ such that each vertex in G is in at least one clique in $\mathcal{C}$, that is,

$$\bigcup_{K \in \mathcal{C}} V(K) = V(G).$$

- A **minimum clique covering** is a clique covering of smallest possible size.
- The **clique covering number** $\theta(G) = \min_{\mathcal{C}} |C|$ is the size of a minimum clique covering.

Examples - domination

- A **dominating set** for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. That is,

$$\bigcup_{v \in D} N[v] = V(G)$$

where $N[v]$ is the closed neighborhood set of v.

- The **domination number** $\gamma(G) = \min_D |D|$ is the number of vertices in a smallest dominating set for G.

Examples - coloring

- A **proper k-vertex coloring** of a graph is a labelling $\{1, 2, \ldots, k\}$ of the vertices $V(G)$ of a graph G with colors such that no two vertices sharing the same edge have the same color. That is,

$$\bigcup_{i=1}^{k} I_i = V(G)$$

where I_i is an independent set.

- A graph is k-**edge colorable** if it has a proper k-edge coloring.
- The **chromatic number** $\chi(G)$ is the least k such that G is k-vertex colorable.

Examples - edge coloring

- A **proper k-edge coloring** of a graph is a labelling $[k]$ of the edges $E(G)$ of a graph G with colors such that no two edges sharing the same vertex have the same color. That is,

$$\bigcup_{i=1}^{k} I_i' = E(G)$$

where I_i' is an independent edge set.

- A **minimum edge covering** is an edge covering of smallest possible size.
- The **chromatic index** $\chi'(G)$ is the least k such that G is k-edge colorable.

Examples - vertex cover

- A **vertex cover** of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The set S is said to "cover" the edges of G, that is

$$\bigcup_{v \in S} N'(v) = E(G)$$

  where $N'(v)$ be the set of edges whose end vertex is v.
- A **minimum vertex cover** is a vertex cover of smallest possible size.
- The **vertex covering number** $\beta(G) = \min_S |S|$ is the size of a minimum vertex cover.

Examples - cycle cover

- An **Euler cycle** is a graph such that the degree of each vertex is even.
- A family $\mathcal{F}$ of Euler cycles of G is a **cycle cover** of G if every edge of G is contained in at least one Euler cycle of $\mathcal{F}$.
- That is,

$$\bigcup_{C \in \mathcal{F}} C = E(G).$$

Examples - cycle cover

- A **minimum cycle cover** is a cycle cover of smallest possible size.
- The **cycle covering number** $c(G) = \min_{\mathcal{F}} |\mathcal{F}|$ is the size of a minimum cycle cover.

Examples - graph process

- Let $f_l : V(G) \to \mathbb{R}$ $(l \geq 0)$ be a labelling of graphs G, then $(f_0, f_1, \dots)$ is called a **graph process**.
- A set $T \subseteq \mathbb{R}$ is called a **target set** of this process.
- The **hitting time** of v is the least number l such that $f_l(v) \in T$.

Examples - graph process

- The **cover time** of G is the least number l such that

$$\bigcup_{v \in V(G)} \{v : f_l(v) \in T\} = V(G).$$

- The n-tuple f_0 is **initial configuration** and f_t is **final configuration** if the process is stop.
- The graph process can be chosen from player or automatically, for example, *chip-firing game* is in the former case and *random walk* on graphs is in the latter case.

Strong edge coloring

- An **induced matching** of a graph is a set of edges in which every two distinct edges are not adjacent and not adjacent to a same edge.
- We say G has a **strong k-edge coloring** if there exists k induced matchings $I_1, I_2, \ldots, I_k$ such that

$$\bigcup_{i=1}^{k} I_i = E(G).$$

- The **strong chromatic index** of G is the least number k such that G has a strong k-edge coloring.

Edge Roman domination

- An **edge Roman dominating function** of a graph G is two disjoint edge subsets (E_1, E_2) of $E(G)$ such that

$$\bigcup_{e \in E_1} e \cup \bigcup_{e \in E_2} N'[e] = E(G)$$

  where $N'[e]$ is the closed neighborhood edge-set of e.

- The **edge Roman domination number** of G is the minimum possible $|E_1| + 2|E_2|$ if (E_1, E_2) is an edge Roman dominating function of G.

Power Roman domination

- An k-**power Roman dominating function** of a graph G is k disjoint vertex subsets $(V_1, \ldots, V_k)$ of $V(G)$ such that

$$\bigcup_{i=1}^{k} \bigcup_{v \in V_i} N_{i-1}[v] = V(G)$$

  where $N_i[v]$ is the set of closed vertex neighbors within distance i.

- The k-**power Roman domination number** of G is the minimum possible $\sum_{i=1}^{k} i|V_i|$ if $(V_1, \ldots, V_k)$ is an k-power Roman dominating function of G.

k-distance edge cover

- An k-**distance edge cover** of a graph G is k disjoint edge subsets $(E_1, \ldots, E_k)$ of $E(G)$ such that

$$\bigcup_{i=1}^{k} \bigcup_{e \in E_i} N_{i-1}(e) = E(G)$$

where $N_i(e)$ is the set of vertex neighbors within distance i.

- The k-**distance edge covering number** of G is the minimum possible $\sum_{i=1}^{k} i|E_i|$ if $(E_1, \ldots, E_k)$ is an k-distance edge cover of G.

Cycle double cover and nowhere-zero flow

- A k-**cycle double cover** is a family of k Euler cycles $C_1, \ldots, C_k$ such that

$$\bigcup_{i=1}^{k} C_i = E(G)$$

and each edge of G is contained in precisely two Euler cycles of this family.

- The **cycle double covering number** of G is the least number k such that G has an k-cycle double cover.

Cycle double cover and nowhere-zero flow

- A **nowhere-zero k-flow** of a graph G is an edge labeling of $\{\pm 1, \ldots, \pm(k-1)\}$ on an orientation $D(G)$ of G such that for every vertex, the sum of in-edge labels is the sum of out edge labels.
- There is an important relation between cycle double cover and nowhere-zero flow, that is, if a graph G has a k-cycle double cover, then G has a nowhere-zero k-flow.

Relaxation procedure

- Suppose G is a connected with $V(G) = \{v_1, v_2, \dots, v_n\}$.
- An n-tuple $X = (x_1, x_2, \dots, x_n) = (X(v_1), \dots, X(v_n))$ of real numbers is called a **configuration** of G if each vertex v_i in G is assigned with the label x_i, and suppose the sum $s = \sum_{i=1}^{n} x_i$ is positive.
- If there is a negative label x_i, then a **legal relaxation** $R^{(i)}$ for X is defined as the operation which transform X into $X' = XR^{(i)} = (x'_1, x'_2, \dots, x'_n)$ obtained from replacing x_i by $-x_i > 0$ and add $2x_i/d_i$ to each of the d_i neighbors of v_i.

Relaxation procedure

- A **relaxation procedure** for X of G is a sequence of configurations $X = X_0, X_1, X_2, \ldots$ and a sequence of relaxations $R^{(k_1)}, R^{(k_2)}, \ldots$ such that $X_i = X_{i-1}R^{(k_i)}$ for $i \geq 1$.
- We say that the relaxation procedure **terminates** if

$$\bigcup_{i=1}^{n} \{v_i : X_t(v_i) > 0\} = V(G)$$

for some t, that is, there is no legal relaxation for X_t.

Primal-dual view

- Consider the linear programming relaxation of the 0-1 integer linear programming of the covering problem.
- For the matrix A and the non-negative vectors $\mathbf{b}, \mathbf{c}$,

$$\begin{aligned} \text{Minimize} \quad &: \quad \mathbf{b}^T\mathbf{x} \\ \text{subject to} \quad &: \quad A^T\mathbf{x} \geq \mathbf{c} \\ & \quad\ \ \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Primal-dual view

- From primal-dual view, we refer to the covering problem as the "primal problem", and we state the "dual problem" as follows.
- For the matrix A and the non-negative vectors $\mathbf{b}, \mathbf{c}$,

$$\begin{aligned} \text{Maximize} \quad &: \quad \mathbf{c}^T \mathbf{y} \\ \text{subject to} \quad &: \quad A\mathbf{y} \le \mathbf{b} \\ & \quad\quad \mathbf{y} \ge \mathbf{0}. \end{aligned}$$

Vertex-version packing

- Let $G = (V, E)$ be a graph.
- Let $\mathcal{P}$ be a property of the vertex set $V(G)$.
- A **vertex-version $\mathcal{P}$-packing** (or simply, **$\mathcal{P}$-packing**) of a graph $G = (V, E)$ is a set $L = \{V_1, \ldots, V_s\}$ of disjoint vertex subsets of $V(G)$, where each vertex subset satisfies the property $\mathcal{P}$.

Vertex-version packing

- Let $\mathcal{P}(\mathcal{P}, G)$ be the set of all $\mathcal{P}$-packing of G.
- Usually the goal is finding the $\mathcal{P}$-**packing number**, that is,

$$\max_{L=\{V_1,\ldots,V_s\}\in\mathcal{P}(\mathcal{P},G)} g(V_1,\ldots,V_s)$$

for some function g.

Edge-version packing

- Similarly, let $\mathcal{P}'$ be a property of the edge set $E(G)$.
- A **edge-version** $\mathcal{P}'$**-packing** (or simply, $\mathcal{P}'$**-packing**) of a graph $G = (V, E)$ is a set $L' = \{E_1, \ldots, E_t\}$ of disjoint edge subsets of $E(G)$, where each edge subset satisfies the property $\mathcal{P}'$.

Edge-version packing

- Let $\mathcal{P}'(\mathcal{P}', G)$ be the set of all $\mathcal{P}'$-packings of G.
- Usually the goal is finding the $\mathcal{P}'$-**packing number**, that is,

$$\max_{L'=\{E_1,\ldots,E_t\}\in\mathcal{P}'(\mathcal{P}',G)} g(E_1,\ldots,E_t)$$

for some function g.

Strong Edge Coloring

G. J. Chang, S.-H. Chen, C.-Y. Hsu, C.-M. Hung and H.-L. Lai,

Strong edge-coloring for jellyfish graphs, submitted.

S.-H. Chen and G. J. Chang,

Perfection for strong edge-coloring on graphs, submitted.

Definitions

- A **strong edge-coloring** of a graph is a function that assigns to each edge a color such that any two edges within distance two apart receive different colors.
- A **color class** of a strong edge-coloring is the set of all edges using the same color.
- A **strong k-edge-coloring** is a strong edge-coloring using at most k colors.

Definitions

- An **induced matching** is an edge set in which two distinct edges are of distance at least two.
- Finding a strong k-edge-coloring is equivalent to partitioning the edge set of the graph into k induced matchings.
- The **strong chromatic index** of a graph G, denoted by $\chi'_s(G)$, is the minimum k such that G admits a strong k-edge-coloring.

Known results

- Strong edge-coloring was first studied by Fouquet and Jolivet (1983) for cubic planar graphs.
- By a greedy algorithm, it is easy to see that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ.
- Fouquet and Jolivet established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam (2009).

Known results

Conjecture

If G is a graph of maximum degree Δ, then $\chi'_s(G) \leq \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.

- It was posed by Erdős and Nešetřil (1989) and revised by Faudree, Gyárfás, Schelp and Tuza (1990).
- For $\Delta = 3$, true by Andersen (1992) and by Horák, Qing and Trotter (1993) independently.
- For $\Delta = 4$, Horák (1990) obtained $\chi'_s(G) \leq 23$ and Cranston (2006) proved $\chi'_s(G) \leq 22$.

Known results

- Molloy and Reed (1997) proved that for large Δ every graph of maximum degree Δ has $\chi'_s(G) \leq 1.998\Delta^2$ using probabilistic method.
- Mahdian (2000) proved that for a C_4-free graph G, $\chi'_s(G) \leq (2 + o(1))\Delta^2/\ln\Delta$.
- Faudree, Gyárfás, Schelp and Tuza (1990) proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \leq \Delta^2$.

Known results

- Brualdi and Massey (1993) improved the upper bound to $\chi'_s(G) \leq \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions.
- Chang and Narayanan (2013) proved that $\chi'_s(G) \leq 8\Delta - 6$ for chordless graphs G.
- Faudree, Gyárfás, Schelp and Tuza (1990) established that $\chi'_s(G) \leq 10\Delta - 10$ for 2-degenerate graphs G.

Known results

Conjecture (Faudree, Gyárfás, Schelp and Tuza, 1990)

$\chi'_s(G) \leq 9$ if G is cubic planar.

- Faudree, Gyárfás, Schelp and Tuza (1990) used the Four-color theorem to show that $\chi'_s(G) \leq 4\Delta(G) + 4$ for any planar graph G of maximum degree Δ.
- They also exhibited a planar graph G whose strong chromatic index is $4\Delta(G) - 4$.

Known results

Conjecture (Faudree, Gyárfás, Schelp and Tuza, 1990)

$\chi'_s(G) \leq 9$ if G is cubic planar.

- They also that $\chi'_s(G) \leq 3\Delta$ for planar graphs G of girth at least 7.
- Chang, Montassier, Pecher and Raspaud (2013) further proved that $\chi'_s(G) \leq 2\Delta - 1$ for planar graphs G with large girth.

Definitions

- A **block** of a graph is a maximal connected subgraph without cut-vertices in itself.
- A **block graph** is a graph whose blocks are complete graphs.
- A **Cactus** is a graph whose blocks are cycles or complete graphs of two vertices.

Trees

- A **closed edge-neighborhood of an edge**

$$\sigma(G) := \max_{uv \in E(G)} (d_G(u) + d_G(v) - 1)$$

is an easy lower bound of $\chi'_s(G)$,

Theorem (Faudree, Gyárfás, Schelp and Tuza, 1990)

$\chi'_s(G) = \sigma(G)$ if G is a tree.

Cycles and complete graphs

Proposition

If $n \geq 3$, then

$$\chi'_s(C_n) = \begin{cases} 5 & \text{if } n = 5; \\ 3 & \text{if } n \text{ is a multiple of } 3; \\ 4 & \text{otherwise.} \end{cases}$$

Proposition

If $n \geq 2$, then $\chi'_s(K_n) = \frac{n(n-1)}{2}$.

Jellyfish graphs

- For a graph H, the H-**jellyfish** $H(p_v\colon v \in V(H))$ is the graph obtained from H by adding p_v new vertices adjacent to v for each vertex v in H.
- An edge which is joining a new vertex to v is called a **pendent edge** at v.

Jellyfish graphs

- A **block-jellyfish** of a graph G is the H-jellyfish H' for some block H of G, where the new vertices of H' are all vertices of $V(G) - V(H)$ having exactly one neighbor in $V(H)$.
- A block-jellyfish is **trivial** if it is an H-jellyfish for an end block H which is K_2, otherwise it is **non-trivial**.

Jellyfish graphs

Theorem

Suppose G is a connected graph that is not a star. If G has exactly r non-trivial block-jellyfishes $G_1, G_2, \ldots, G_r$, then

$$\chi_s'(G) = \max_{1 \le i \le r} \chi_s'(G_i).$$

Corollary

If G is a block graph, then $\chi_s'(G) = \max\{|E(H)| \colon H$ is a non-trivial block-jellyfish of $G\}$.

Theorem

If G is a C_n-jellyfish of m edges with $\sigma(G) \geq 4$, then $\chi'_s(G) =$

$$\begin{cases} m, & \text{if } n = 3; \\ \sigma(G)+1, & \text{if } n = 4; \\ \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil, & \text{o.w., if } n \text{ is odd with all } d_G(v_i) = d \text{ but } (n,d) \neq (7,3), \\ & \quad \text{or } \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \geq \sigma(G)+1; \\ \sigma(G)+1, & \text{o.w., if } (n,d) = (7,3) \text{ with all } d_G(v_i) = d, \\ & \quad \text{or } n \not\equiv 0 (\bmod\ 3) \text{ such that up to rotation} \\ & \quad\quad d_G(v_i) = \sigma(G) - 1 \text{ for } i \equiv 1\ (\bmod\ 3) \\ & \quad\quad \text{with } 1 \leq i \leq 3\lfloor \frac{n}{3} \rfloor - 2, \\ & \quad \text{or } (n, \sigma(G)) = (10, 4) \\ & \quad\quad \text{with } d_G(v_i) = 3 \text{ for all odd or all even } i; \\ \sigma(G), & \text{o.w..} \end{cases}$$

Definitions

- An **anti-matching** is a set of edges in which every two distinct edges are adjacent or are adjacent to a same edges.
- The **closed edge-neighborhood of a clique** C is the set $N_e[C] = \{e \in E \colon e$ is incident to some vertex in $C\}$.
- The **closed edge-neighborhood of an edge** xy is the set $N_e[xy] = \{e \in E \colon e$ is incident to x or $y\}$.

Definitions

- The **anti-matching number** am(G) of a graph G is the maximum size of an anti-matching.
- $\sigma^*(G) := \max_{C\text{:clique}} |N_e[C]| = \max_{C\text{:clique}} \left(\sum_{x \in C} d_G(x) - \binom{|C|}{2} \right)$
- $\sigma(G) := \max_{xy \in E} |N_e[xy]| = \max_{xy \in E} (d_G(x) + d_G(y) - 1)$.

Weak duality inequalities

- $\sigma(G) \leq \sigma^*(G) \leq \mathrm{am}(G) \leq \chi'_s(G)$.

Lemma

If G has no clique of size 3 in which each vertex is of degree at least 3, then $\sigma(G) = \sigma^*(G)$.

Theorem (Liao, 2012)

If G is a cactus in which the length of a cycle is a multiple of 6, then $\chi'_s(G) = \sigma(G)$.

Corollary (Faudree, Gyárfás, Schelp and Tuza, 1990)

If T is a tree, then $\chi'_s(T) = \sigma(T)$.

Weak duality inequalities

- $\sigma(G) \leq \sigma^*(G) \leq \mathrm{am}(G) \leq \chi'_s(G)$.

Theorem (Cameron, 1989)

If G is chordal, then $\chi'_s(G) = \sigma^*(G)$.

- A graph G is **weakly chordal** if neither the graph nor its complement contains an induced cycle of length at least five in G.

Theorem (Cameron, Sritharan and Tang, 2003)

If G is weakly chordal, then $\chi'_s(G) = am(G)$.

Perfection

- For any $t \in \{\sigma, \sigma^*, \text{am}\}$, a graph G is **vertex t-perfect** (respectively, **edge t-perfect**) if $t(H) = \chi'_s(H)$ for any induced (respectively, edge-induced) subgraph H of G.
- See the following flowchart for a summary.

$$\begin{array}{ccccc} \text{edge } \sigma\text{-perfect} & \Rightarrow & \text{edge } \sigma^*\text{-perfect} & \Rightarrow & \text{edge am-perfect} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{vertex } \sigma\text{-perfect} & \Rightarrow & \text{vertex } \sigma^*\text{-perfect} & \Rightarrow & \text{vertex am-perfect} \end{array}$$

Perfection

Corollary

If G is a cactus in which the length of a cycle is a multiple of 6 or a forest, then G is edge t-perfect and vertex t-perfect for $t \in \{\sigma, \sigma^*, am\}$.

Corollary

Chordal graphs are vertex σ^*-perfect and vertex am-perfect.

Perfection

Corollary

Chordal graphs without any clique of size 3 in which each vertex is of degree at least 3 are vertex t-perfect for $t \in \{\sigma, \sigma^*, am\}$.

Corollary

Weakly chordal graphs are vertex am-perfect.

Graphs with cycle lengths of multiple 3

Theorem

If G is a 2-connected graph in which the length of every cycle is a multiple of 3, then $\chi'_s(G) = \sigma(G)$.

Conjecture

2-connected graphs in which the length of every cycle is a multiple of 3 are vertex σ-perfect.

Edge Roman Domination

G. J. Chang, S.-H. Chen and C.-H. Liu,

Edge Roman domination on graphs, submitted.

Roman domination

- The articles by ReVelle (1997) in the Johns Hopkins Magazines suggested a new variation of domination called Roman domination
- A region of the Roman empire is considered to be *unsecured* if it has no mobile Field Armies (FA) stationed there and *secured* otherwise.

Roman domination

- In the 4th century A.D., Constantine the Great (Emperor of Rome) issued a decree that a FA cannot be sent from a secured region to an unsecured region if doing so leaves that region unsecured.
- Thus, there are two types of armies, stationary and traveling.

Roman domination

- Each vertex (city) has no army must have a neighboring vertex with a traveling army.
- Stationary armies then dominate their own vertices, and a vertex with two armies is dominated by its stationary army, and its open neighborhood is dominated by the traveling army.

Roman domination

- We may formulate the problem in terms of graphs.
- A **Roman dominating function** of a graph G is a function $f\colon V(G) \to \{0,1,2\}$ such that every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$.

Roman domination

- The **weight** of a Roman dominating function f is the value $w(f) = \sum_{v \in V(G)} f(v)$.
- The **Roman domination number** of G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function of G.

Edge Roman domination

- Recently, Roushini Leely Pushpam and Malini Mai (2009) initiated the study of the edge version of Roman domination.
- An **edge Roman dominating function** of a graph G is a function $f\colon E(G) \to \{0,1,2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$.

Edge Roman domination

- The **weight** of an edge Roman dominating function f is the value $w(f) = \sum_{e \in E(G)} f(e)$.
- The **edge Roman domination number** of G, denoted by $\gamma'_R(G)$, is the minimum weight of an edge Roman dominating function of G.

Edge Roman domination

- In fact, the edge Roman domination number of G equals the Roman domination number of its line graph.
- However, we are interesting in finding upper bound of $\gamma'_R(G)$ in terms of $|V(G)|$ instead of $|E(G)|$.

Known results

- Roushini Leely Pushpam *et al.* (2009):

$$\gamma'_R(P_n) = \lfloor \frac{2n}{3} \rfloor \text{ and } \gamma'_R(C_n) = \lceil \frac{2n}{3} \rceil.$$

- Akbari *et al.*: $\gamma'_R(G) \leq \frac{2\Delta}{2\Delta+1} n$.

Conjecture (Δ-conjecture, Akbari *et al.*)

If G is a graph of maximum degree Δ on n vertices, then

$\gamma'_R(G) \leq \lceil \frac{\Delta}{\Delta+1} n \rceil$.

Known results

Akbari, Ehsani, Ghajar, Khalilabadi and Sadeghabad:

- If G has a perfect matching, then $\gamma'_R(G) \leq \frac{2\Delta-1}{2\Delta}n$.
- If T is a tree of n vertices, then

$$\lceil \frac{2(n-\ell(T)+1)}{3} \rceil \leq \gamma'_R(T) \leq \lceil \frac{2(n-1)}{3} \rceil = \lfloor \frac{2n}{3} \rfloor$$

where $\ell(T)$ is the number of leaves, and the equality holds if and only if $T = P_n$.

Known results

Akbari, Ehsani, Ghajar, Khalilabadi and Sadeghabad:

- If $n \geq 2$, then $\gamma'_R(P_2 \Box P_n) = \lceil \frac{4n}{3} \rceil$ and $\gamma'_R(P_3 \Box P_n) = 2n$.
- If $n \geq 1$, then $\gamma'_R(Q_n) \geq \frac{2^{n+1}n}{3n-1}$.

Known results

Akbari and Qajar:

- If G is outerplanar, then $\gamma'_R(G) \leq \frac{4}{5}n$.
- If G is planar and claw-free, then $\gamma'_R(G) \leq \frac{6}{7}n$.

Conjecture

If G is a planar graph of n vertices, then $\gamma'_R(G) \leq \frac{6}{7}n$.

Known results

Theorem

If $1 \le r \le s$, then $\gamma'_R(K_{r,s}) = 2r$ for $r < s$ and $\gamma'_R(K_{r,s}) = 2r - 1$ for $r = s$.

- $\gamma'_R(K_{r,r}) = 2r - 1 = \frac{2\Delta-1}{2\Delta}n$, while the gap between $\frac{2\Delta-1}{2\Delta}n$ and $\frac{\Delta}{\Delta+1}n$ being $\frac{\Delta-1}{2\Delta(\Delta+1)}n$.
- $\gamma'_R(K_{r,r+1}) = 2r = \frac{2\Delta-2}{2\Delta-1}n$, while the gap between $\frac{2\Delta-2}{2\Delta-1}n$ and $\frac{\Delta}{\Delta+1}n$ is $\frac{\Delta-2}{(\Delta+1)(2\Delta-1)}n$.

$G_{r,t}$

- Consider the graph $G_{r,t}$ obtained from t copies of $K_{r,r+1}$ by adding edges $y^i_{r+1}y^{i+1}_1$ for $1 \leq i \leq t$ with $y^{t+1}_1 = y^1_1$, where the partite sets of the i-th $K_{r,r+1}$ are $X_i = \{x^i_1, x^i_2, \ldots, x^i_r\}$ and $Y_i = \{y^i_1, y^i_2, \ldots, y^i_{r+1}\}$.
- To get counterexamples, we modify complete bipartite graphs to obtain graphs whose Δ are far away from n.

$G_{r,t}$

- $G_{2,4}$

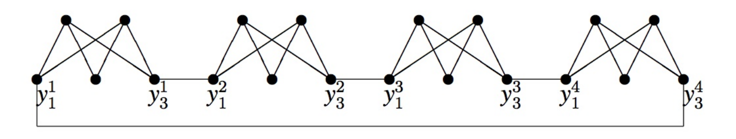


Figure : The graph $G_{2,4}$.

Counterexamples to Δ-conjecture

Conjecture (Δ-conjecture, Akbari *et al.*)

If G is a graph of maximum degree Δ on n vertices, then $\gamma'_R(G) \leq \lceil \frac{\Delta}{\Delta+1} n \rceil$.

- $\gamma'_R(G_{r,t}) = 2rt = \frac{2\Delta-2}{2\Delta-1} n > \frac{\Delta}{\Delta+1} n = \lceil \frac{\Delta}{\Delta+1} n \rceil$ when $r \geq 2$ and t a multiple of $r+2$.
- This disproves Δ-conjecture.

Removable triple

- A **removable triple** of a graph G is a triple (S, M_2, M_1), where S is a nonempty subset of $V(G)$ and M_2 and M_1 are disjoint matchings in $G[S]$ such that every edge $e \in E(G) - M_1$ incident to a vertex in S is adjacent to some edge in M_2.
- We define the *ratio* $\rho(S, M_2, M_1)$ of a removable triple (S, M_2, M_1) to be $\frac{2|M_2|+|M_1|}{|S|}$.

Removable triple

Lemma

If a graph G has a removable triple (S, M_2, M_1) with $\rho(S, M_2, M_1) \leq \alpha$, then $\gamma'_R(G) \leq \gamma'_R(G - S) + \alpha|S|$.

Lemma

For every removable triple (S, M_2, M_1) of G, if $\gamma'_R(G - S) \leq \alpha|V(G - S)|$ but $\gamma'_R(G) > \alpha|V(G)|$, then $\rho(S, M_2, M_1) > \alpha$

Removable triple

Lemma

If v is a vertex of degree d in a graph G and M is a matching in $G[N(v)]$, then G has a removable triple (S, M_2, M_1) with $|S| \leq 2d + 1$ and

$$\rho(S, M_2, M_1) \leq \frac{2d - 2|M|}{2d + 1 - 2|M|} \leq \frac{2d}{2d + 1}.$$

k-degenerate graph

Theorem

If G is a k-degenerate graph of n vertices, then $\gamma'_R(G) \leq \frac{2k}{2k+1}n$.

Corollary

If T is a tree of n vertices, then $\gamma'_R(T) \leq \frac{2}{3}n$.

Corollary

If G is a outerplanar graph of n vertices, then $\gamma'_R(G) \leq \frac{4}{5}n$.

k-degenerate graph

Lemma

Let G be a graph of maximum degree Δ of n vertices. If every component of G contains a vertex of degree less than Δ, then $\gamma'_R(G) \leq \frac{2\Delta-2}{2\Delta-1}n$.

Theorem

If G is a connected graph of maximum degree Δ on n vertices, then $\gamma'_R(G) \leq \frac{2\Delta-2}{2\Delta-1}n + \frac{2}{2\Delta-1}$.

Subcubic graphs

- Recall that Akbari showed that $\gamma'_R(G) \leq \frac{6}{7}n$ for every subcubic graph G of n vertices.

Theorem

If G is a subcubic graph of n vertices contains no $K_{3,3}$ as a component, then $\gamma'_R(G) \leq \frac{4}{5}n$.

Graphs on surfaces of small genus

Conjecture (Akbari and Qajar)

If G is a planar graph of n vertices, then $\gamma'_R(G) \leq \frac{6}{7}n$.

Theorem

If G is a graph of n vertices that can be embedded in the plane or the projective plane, then $\gamma'_R(G) \leq \frac{6}{7}n$.

Graphs on surfaces of small genus

Sketch of proof:

- Every vertex of G has degree at least four.
- Discharging method: For every $x \in V(G) \cup F(G)$, we define the charge ch(x) on x to be $\deg(x) - 4$.
- According to Euler's formula, the sum of the charge is

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -4|V| + 4|E| - 4|F| < 0.$$

Graphs on surfaces of small genus

Sketch of proof:

- For every vertex v incident to exactly t 3-faces with $t > 0$, we move $\frac{\deg(v)-4}{t}$ units of charge to each 3-face incident to it.
- We denote the new charge on each $x \in V(G) \cup F(G)$ by $\mathrm{ch}'(x)$. Clearly, $\sum_{x \in V(G) \cup F(G)} \mathrm{ch}(x) = \sum_{x \in V(G) \cup F(G)} \mathrm{ch}'(x)$.
- We shall prove that $\mathrm{ch}'(x) \geq 0$ for every $x \in V(G) \cup F(G)$.

Graphs on surfaces of small genus

Theorem

Let Σ be the plane, projective plane, torus or Klein bottle. If G is a graph of girth at least 5 on n vertices that can be embedded in Σ, then $\gamma'_R(G) \leq \frac{4}{5}n$.

Conjecture

If G is a planar graph of girth at least $3k+2$ on n vertices, then $\gamma'_R(G) \leq \frac{2k+2}{3k+2}n$.

Graph without $K_{2,3}$-subdivisions

Lemma

A graph G is an outerplanar graph if and only if G does not contain a subgraph isomorphic to a subdivision of K_4 or $K_{2,3}$.

Theorem

Let G be a graph of n vertices that does not contain a subgraph isomorphic to a subdivision of $K_{2,3}$. If G does not contain C_5 as a component and there does not exist a vertex v such that $G - v$ contains C_5 as a component, then $\gamma'_R(G) \leq \frac{3}{4}n$.

Power Roman Domination

S.-H. Chen, J. K. Lan and G. J. Chang,

Power roman domination in graphs, submitted.

S.-H. Chen, J. K. Lan and G. J. Chang,

Algorithmic aspects of Power Roman Domination in Graphs,

submitted.

Roman domination

- Recall that Roman domination in graphs is a variation of domination suggested implicitly by (1997) and Stewart (1999).
- A *Roman dominating function* (RDF) of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has a neighbor u with $f(u) = 2$.
- The *weight* of f is $w(f) = \sum_{v \in V(G)} f(v)$.

Roman domination

- The *Roman domination number* of G, $\gamma_R(G)$, is the minimum of $w(f)$ over all such functions.
- The *Roman domination problem* is to determine an RDF of weight $\gamma_R(G)$ of graph G.

Roman domination

Cockayne, Dreyer, Hedetniemi and Hedetniemi:

- $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$, and that $\gamma(G) = \gamma_R(G)$ if and only if G has no edges, where $\gamma(G)$ is the domination number of G.
- $\gamma_R(G) \leq \frac{2n}{\delta(G)+1}(\ln \frac{\delta(G)+1}{2} + 1)$ for a graph G of order n and minimum degree $\delta(G)$.

Roman domination

- Chambers *et al.* (2009) proved that $\gamma_R(G) \leq \frac{4n}{5}$ when $n \geq 3$ and $\delta(G) \geq 1$, and $\gamma_R(G) \leq \frac{8n}{11}$ when $n \geq 9$ and $\delta(G) \geq 2$.
- Liu and Chang (2012) showed that $\gamma_R(G) \leq \frac{2n}{3}$ when $\delta(G) \geq 3$ and $\gamma_R(G) \leq \max\{\lceil \frac{2n}{3} \rceil, \frac{23n}{34}\}$ when G is 2-connected.

Roman domination

- Liedloff *et al.* (2008) presented linear-time algorithms on interval graphs and on co-graphs, and established a polynomial-time algorithm on AT-free graphs.
- Liu and Chang (2013) proposed a linear-time algorithm for the weighted (a, b)-Roman domination problem with $b \geq a > 0$ on strongly chordal graphs, and showed that the decision version of the Roman domination problem is NP-complete on bipartite graphs and on split graphs.

power Roman domination

- We assume that a vertex v with $i(>0)$ FAs stationed can **power-dominate** all the vertices that have distance within $i-1$ from v.
- More specifically, for a fixed positive integer $k \geq 2$, a k-**power Roman dominating function** (kPRDF) of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots, k\}$ such that for every vertex v with $f(v) = 0$, there exists a vertex u with $f(u) > 0$ and the distance between u and v in G is less than $f(u)$.

power Roman domination

- The **weight** of f is $w(f) = \sum_{v \in V(G)} f(v)$.
- The k-**power Roman domination number** of G, $\gamma_{PR,k}(G)$, is the minimum of $w(f)$ over all such functions.
- The k-**power Roman domination problem** is to determine a kPRDF of weight $\gamma_{PR,k}(G)$ of graph G.

power Roman domination

- The special case when $k = 2$ is the ordinary Roman domination.
- In other words, $\gamma_{PR,2}(G) = \gamma_R(G)$ for any graph G.

Properties

- A **universal vertex** is a vertex adjacent to all other vertices in the graph.

Proposition

Let G be a graph of order at least 2. Then $\gamma_{PR,k}(G) = 2$ if and only if $G = 2K_1$ or has a universal vertex, for any integer $k \geq 2$.

Properties

Proposition

Let $f = (V_0, V_1, V_2, \ldots, V_k)$ be a $\gamma_{PR,k}(G)$-function. Then

1. $G[V_i]$ is an independent set for all $i \geq 2$ with $i + 1 \leq k$.
2. No edge of G joins V_1 and V_i for all $i \geq 2$.
3. Each vertex of V_0 is adjacent to at most two vertices of V_1.
4. $V_0 \cup V_2 \cup V_3 \cup \cdots \cup V_k$ is a γ-set of $G^{k-1}[V_0 \cup V_2 \cup V_3 \cup \cdots \cup V_k]$.

Properties

Proposition

Let $f = (V_0, V_1, \ldots, V_k)$ be a $\gamma_{PR,k}(G)$-function that maximizes $|V_0|$.

1. V_1 is an independent set.
2. $V_0 \cup V_2 \cup V_3 \cup \cdots \cup V_k$ is a vertex cover of G.
3. Each vertex of V_0 is adjacent to at most one vertex of V_1, i.e., V_1 is a packing.
4. V_0 is a dominating set of $G[V_0 \cup V_1]$ if G has no isolated vertices.
5. If $u \in V_i, v \in V_j$ for some $i, j \in \{1, 2, \ldots, k-1\}$ and $i + j \leq k$, then $d_G(u, v) > i + j$.
6. If $u \in V_i, v \in V_j$ for some $i, j \in \{1, 2, \ldots, k-1\}$ $i + j > k$, then $d_G(u, v) > 2k - i - j$.

General bounds

Proposition

For any graph G and a spanning subgraph H of G,

$$\gamma_{PR,k}(G) \leq \gamma_{PR,k}(H)$$

for any positive integer k.

Theorem

For any connected graph G of order n and rad$(G) = r$,

$$r + 1 = \cdots = \gamma_{PR,r+1}(G) \leq \cdots \leq \gamma_{PR,2}(G) \leq \gamma_{PR,1}(G) = n.$$

General bounds

Proposition (Cockayne *et al.*, 2004)

For any graph G, $\gamma(G) \leq \gamma_{PR,2}(G) \leq 2\gamma(G)$.

Proposition

For any graph G,

$$\gamma(G^{k-1}) \leq \gamma_{PR,k}(G) \leq 2\gamma(G)$$

for any integer $k \geq 3$. Moreover, $\gamma(G^{k-1}) = \gamma_{PR,k}(G)$ if and only if $G = \overline{K_n}$.

General bounds

Proposition (Cockayne *et al.*, 2004)

For any graph G of order n and $\Delta(G) = \Delta$,

$$\frac{2n}{\Delta + 1} \leq \gamma_{PR,2}(G).$$

Theorem

For any graph G of order n and $\Delta(G) = \Delta \geq 3$,

$$\frac{(\Delta - 2)kn}{\Delta(\Delta - 1)^{k-1} - 2} \leq \gamma_{PR,k}(G)$$

for any integer $k \geq 2$. Moreover, the inequality holds if and only if G has a disjoint union of full Δ-ary trees as a spanning subgraph.

General bounds

Proposition (Cockayne *et al.*, 2004)

For a graph G of order n,

$$\gamma_{PR,2}(G) \leq \frac{2 + 2\ln((1+\delta(G))/2)}{1+\delta(G)} n.$$

Theorem

For any graph G of order n and $\delta_k(G) = \delta_k$,

$$\gamma_{PR,k}(G) \leq \frac{k}{\delta_{k-1}+1} \left(\ln \frac{\delta_{k-1}+1}{k} + 1 \right) n$$

for any integer $k \geq 2$.

Complete multipartite graphs

Proposition

For the complete graph K_n,

$$\gamma_{PR,k}(K_n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n \geq 2, \end{cases}$$

for any integer $k \geq 2$.

Proposition

For the complete multipartite graph $K_{r_1,r_2,\ldots,r_s}$, $s \geq 2$, $\gamma_{PR,k}(K_{r_1,r_2,\ldots,r_s}) = \min\{r_1+1, r_2+1, \ldots, r_s+1, 4\}$ for any integer $k \geq 2$.

Paths and cycles

Proposition

For the path P_n and the cycle C_n,

$$\gamma_{PR,k}(P_n) = \gamma_{PR,k}(C_n) = \left\lceil \frac{kn}{2k-1} \right\rceil$$

for any positive integer k.

$2 \times n$ grid graphs

Proposition

For the $2 \times n$ grid graph $G_{2,n}$,

$$\gamma_{PR,3}(G_{2,n}) = \begin{cases} 3, & \text{if } n = 3; \\ n + 1 - \left\lfloor \frac{n}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$

$2 \times n$ grid graphs

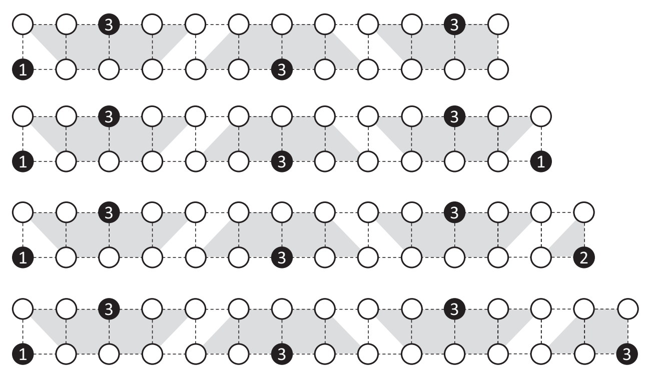


Figure : Power Roman domination on grid graphs $G_{2,n}$; $12 \leq n \leq 15$, $k = 3$.

Cartesian products of two paths or two cycles

Theorem

For the Cartesian products of two paths or two cycles,

$$\lim_{m,n\to\infty} \frac{\gamma_{PR,k}(P_m \Box P_n)}{mn} = \lim_{m,n\to\infty} \frac{\gamma_{PR,k}(C_m \Box C_n)}{mn} = \frac{k}{2k^2 - 2k + 1}$$

for any positive integer k

Proposition

For any positive integers k, m and n,

$$\gamma_{PR,k}(C_{(2k^2-2k+1)m} \Box C_{(2k^2-2k+1)n}) = (2k^2 - 2k + 1)kmn.$$

Cartesian products of two paths or two cycles

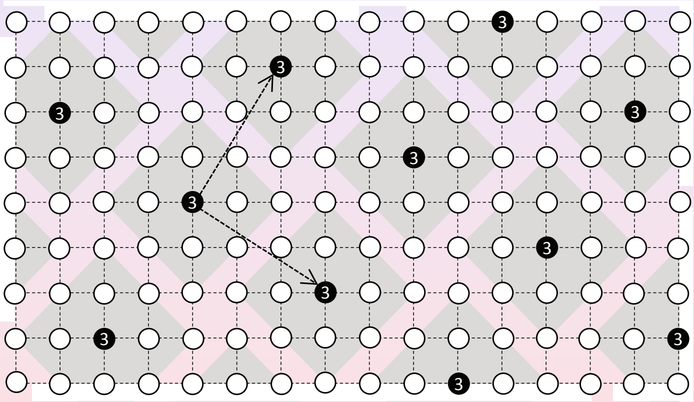


Figure : Power Roman domination on a grid graph; $k = 3$

Trees

Theorem (Chambers, 2009)

For a tree T of order $n \geq 3$, $\gamma_{PR,2}(T) \leq \frac{4n}{5}$.

Theorem

For an positive integer $k \geq 2$ and a tree $T \neq P_{2\ell}$ for some $\left\lfloor \frac{k+1}{3} \right\rfloor + 1 \leq \ell \leq \left\lfloor \frac{2k+1}{3} \right\rfloor$ of order $n \geq 2\left\lfloor \frac{k+1}{3} \right\rfloor + 1$,

$$\gamma_{PR,k}(T) \leq \frac{k+2}{2k+1} n.$$

Trees

Corollary

For a tree T of order $n \geq 3$ with $T \neq P_4$,

$$\gamma_{PR,3}(T) \leq \frac{5}{7}n.$$

Corollary

For any connected graph G of order $n \geq 3$ with $G \neq P_4$,

$$\gamma_{PR,3}(G) \leq \frac{5}{7}n.$$

NP-completeness results

k-POWER ROMAN DOMINATION (kPRD)

INSTANCE: A graph $G = (V, E)$ and positive integers k and s.

QUESTION: Does G have a kPRDF of weight $\leq s$?

Theorem

For any fixed integer $k \geq 2$, $k\mathrm{PRD}$ is NP-complete on chordal graphs.

NP-completeness results

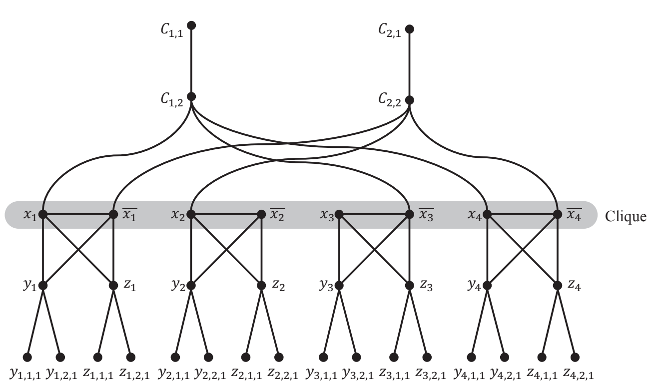


Figure : kPRD instance in chordal graph, resulting from 3-SAT instance; $k = 3$.

NP-completeness results

Theorem

For any fixed integer $k \geq 3$, $k\mathrm{PRD}$ is NP-complete on planar bipartite graphs.

Corollary

For any fixed integer $k \geq 3$, $k\mathrm{PRD}$ is NP-complete on planar graphs.

Corollary

For any fixed integer $k \geq 3$, $k\mathrm{PRD}$ is NP-complete on bipartite graphs.

NP-completeness results

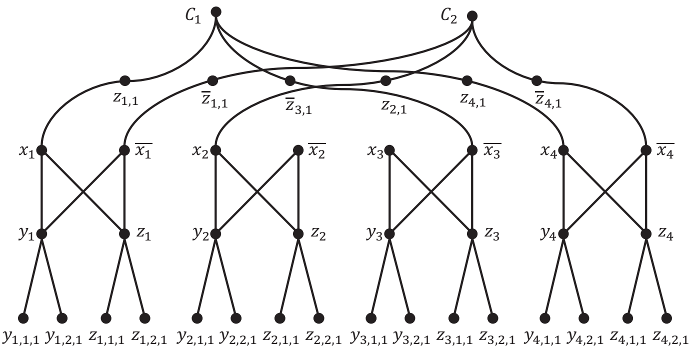


Figure : kPRD instance in planar bipartite graph, resulting from 3-SAT instance; $k = 3$.

Strongly chordal graphs

- A **strong elimination ordering** of a graph $G = (V, E)$ is an ordering $v_1, v_2, \ldots, v_n$ of V such that

$$a \leq b, c \leq d, a \sim c, a \sim d \text{ and } b \sim c \text{ imply } b \sim d.$$

- We now generalize this elimination ordering involving the "distance neighborhood".

Strongly chordal graphs

- Given a fixed positive integer k, a k-**universal elimination ordering** is an ordering $v_1, v_2, \ldots, v_n$ of V such that

$$a \leq b, c \leq d, a \overset{j}{\sim} c, a \overset{j}{\sim} d \text{ and } b \overset{j}{\sim} c \text{ imply } b \overset{j}{\sim} d \qquad (k\text{-UEO})$$

  for all integers $1 \leq j \leq k$, where the notation $a \overset{j}{\sim} b$ means $v_a \in N_G^j[v_b]$. In other words, a k-UEO for G is a SEO for G^j for each $1 \leq j \leq k$.

Strongly chordal graphs

- For a fixed positive integer k, let $\mathcal{G}_k$ denote the set of all graphs that admit a k-UEO.
- Let *SC* denote the set of all strongly chordal graphs.

Theorem

For any fixed positive integer k, $\mathcal{G}_k = SC$.

Algorithm on strongly chordal graphs

(LSR)

minimize $\sum_{\ell=1}^{k}\sum_{i=1}^{n} \ell\omega_i x_{\ell,i},$

subject to $x_{\ell,i} \geq 0 \quad \forall\, 1 \leq \ell \leq k,\ \forall\, 1 \leq i \leq n,$ (LSR1)

$$\sum_{\ell=1}^{k} \sum_{v_i \in N_G^{\ell-1}[v_j]} x_{\ell,i} \geq 1\ \forall\, j. \quad \text{(LSR2)}$$

Algorithm on strongly chordal graphs

(DSR)

maxmize $\displaystyle\sum_{j=1}^{n} y_j,$

subject to $\displaystyle\sum_{v_j \in N_G^{\ell-1}[v_i]} y_j \leq \ell w_i \;\forall\, 1 \leq \ell \leq k, 1 \leq i \leq n,$ (DSR1)

$y_j \geq 0$ for all j. (DSR2)

Algorithm on strongly chordal graphs

Algorithm **PRSC** (Power Roman domination for strongly chordal graphs)

Input: A strong chordal graph G with a $(k-1)$-strong elimination ordering $v_1, v_2, \ldots, v_n$ and nonnegative vertex weights $w_1, w_2, \ldots, w_n$.

Output: Optimal solution $x_{l.i}, y_j$ to (LSR) and (DSR) where $(x_{l,i})$'s are mutually orthogonal $(0,1)$-vectors.

stage 1: each $x_{l,i} \leftarrow 0$, each $y_j \leftarrow 0$; unmark all v_i;

stage 2: **for** $j = 1$ **to** n **do**

$$y_j \leftarrow \min\{w_j, lw_i - \sum_{v_s \in N_{l-1}[v_i]} y_s : v_i \in N_{l-1}[v_j], 2 \le l \le k\};$$

Algorithm on strongly chordal graphs

Algorithm **PRSC** (Power Roman domination for strongly chordal graphs)

stage 3: **for** $i = n$ **to** 1 **step** -1 **do**

for $l = k$ **to** 2 **step** -1 **do**

if $lw_i = \sum_{v_s \in N_{l-1}[v_i]} y_s$ and there is no marked $v_s \in N_{l-1}[v_i]$ with $y_s > 0$

then $x_{l,i} \leftarrow 1$ and mark all vertices in $N_{l-1}[v_i]$, **berak**;

stage 4: **for** $j = 1$ **to** n **do**

if v_j is unmarked **then** $x_{1,j} \leftarrow 1$.

Algorithm on strongly chordal graphs

Theorem

Algorithm $\mathbf{PRSC}$ *finds the minimum weighted* k*-power Roman domination number of a strongly chordal graph* G *with nonnegative vertex weights in time* $O(kn\Delta^k(G))$ *with a* $(k-1)$*-universal elimination ordering provided.*

k-Distance Edge Cover

S.-H. Chen, J. K. Lan, L.-H. Huang and G. J. Chang,

k-Distance edge cover on Graphs, submitted.

Edge cover

- The **distance** $\text{dist}(e, v)$ between an edge e and a vertex v in a graph is the length of a shortest path between them.
- The **vertex-neighbors within distance** k denotes

$$N_k(e) = \{v : \text{dist}(e, v) \leq k\}.$$

Edge cover

- An **edge cover** of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set.
- That is, $C \subseteq E$ is an edge cover if

$$\bigcup_{e \in C} N_0(e) = V(G).$$

- The **edge covering number** $\beta'(G)$ is the minimum size of edge cover.

k-distance Edge cover

- Next, we would generalize the definition, the edge partition $C = (E_0, E_1, \ldots, E_k)$ is an k-**distance edge cover** if

$$\bigcup_{i=1}^{k} \Big(\bigcup_{e \in E_i} N_{i-1}(e) \Big) = V(G).$$

- It is easy to see that an edge cover is an 1-distance edge cover.

k-distance Edge cover

- The **weight** of an k-distance edge cover is the sum $w(C) = \sum_{i=1}^{k} i|E_i|$.
- The minimum weight is called an k-**distance edge covering number** $\beta'_k(G)$.

General properties

Proposition

For any graph G, if H is a spanning subgraph of G, then $\beta'_k(G) \leq \beta'_k(H)$ for each k.

Proposition

If G has a perfect matching M, then $\beta'_2(G) \leq \frac{|V(G)|}{2}$.

Proposition

If M is a matching of G, then $\beta'_2(G) \leq n - |M|$.

General properties

Proposition

For any $k \geq 1$, $\beta'_k(G) \geq 2$ if $|E(G)| \geq 2$.

Proposition

For any $k \geq 2$. Let G be a graph with $|E(G)| \geq 2$, $\beta'_k(G) = 2$ if and only if $G = 2K_2$ or there exists an edge xy such that $N(x) \cup N(y) = V(G)$.

General Inequality

- The **edge-vertex-diameter** $\mathrm{diam}'(G)$ is

$$\max_{e \in E(G), v \in V(G)} \mathrm{dist}(e, v).$$

- The **vertex-eccentricity** of an edge e, written $\mathrm{ecc}'(e)$, is

$$\max_{v \in V(G)} \mathrm{dist}(e, v)$$

- The **edge-vertex-radius** of a graph G, written $\mathrm{rad}'(G)$, is

$$\min_{e \in E(G)} \mathrm{ecc}'(e).$$

General Inequality

Theorem

Let r' be the edge-vertex-radius of a connected graph G, then

$$\beta'_1(G) \geq \beta'_2(G) \geq \cdots \geq \beta'_{r'+1}(G) = \beta'_{r'+2}(G) = \cdots = r' + 1.$$

Lower bounds

Theorem

For any graph G of order n and maximum degree $\Delta \geq 3$,

$$\beta'_k(G) \geq \frac{(\Delta - 2)kn}{2(\Delta - 1)^k - 2}.$$

Cycles, paths and complete graphs

Proposition

For any $k \geq 1$, $\beta'_k(P_n) = \beta'_k(C_n) = \lfloor \frac{n+1}{2} \rfloor$.

Proposition

For any $k \geq 2$, $\beta'_k(K_n) = 2$ if $n \geq 3$.

Complete multipartite graphs

Proposition

For any $k \geq 2$, $\beta'_k(K_{m,n}) = 2$ if $(m, n) \neq (1, 1)$.

Proposition

For any $k \geq 2$, $\beta'_k(K_{r_1,r_2,\ldots,r_s}) = 2$ if $s > 3$ or $\max_i r_i > 1$.

Trees

- Let $J_{a,b}$ be a graph which is formed by a root vertex r adjacent with a isolated vertices and b claws (connect to the leave of each claw).

Lemma

If one component of $T - e$ is an odd path or isolated vertex for any edge $e \in E(T)$, then either T is a path with five vertices, a claw adding three pendent vertices on its leaves, or a graph $J_{a,b}$ for some nonnegative a, b.

Trees

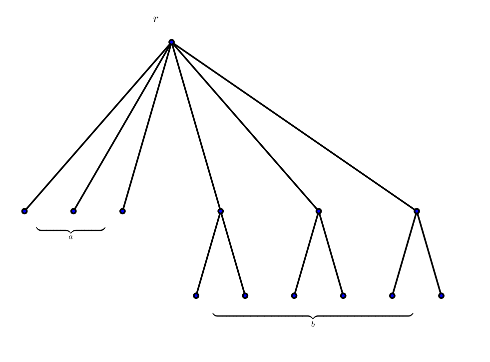


Figure : The graph $J_{a,b}$.

Trees

Theorem

For $k \geq 3$, if T is not an odd path, then $\beta'_k(T) \leq \frac{n(T)}{2}$.

Theorem

For $k = 2$, $\beta'_2(T) \leq \frac{2}{3}n(T)$ and the equation holds if and only if $T = P_3$.

Cartesian product of graphs

Theorem

For $k \geq 1$, $\lim_{m,n\to\infty} \frac{\beta'_k(P_m \square P_n)}{mn} = \lim_{m,n\to\infty} \frac{\beta'_k(C_m \square C_n)}{mn} = \frac{1}{2k}$.

Theorem

For $k \geq 1$, $\beta'_k(C_{2km} \square C_{2kn}) = 2kmn$.

NP-completeness results

k-DISTANCE EDGE COVER

INSTANCE: A graph $G = (V, E)$ and positive integers $k, s \leq |V|$.

QUESTION: Does G have an k-distance edge cover C with $w(C) \leq s$?

Theorem

For any $k \geq 2$, the decision problem k-DISTANCE EDGE COVER is NP-complete for chordal graphs.

NP-completeness results

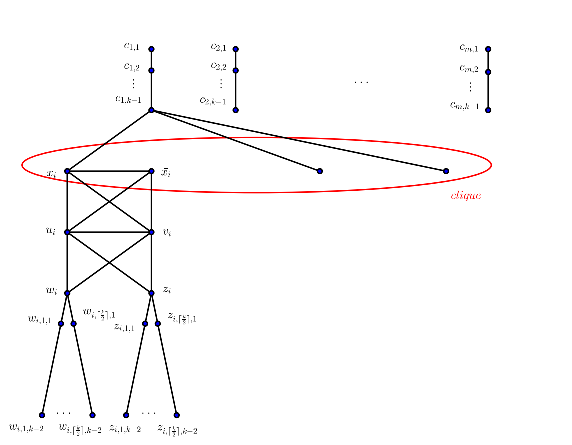


Figure : A transformation to a chordal graph.

NP-completeness results

Theorem

For any $k \geq 2$, the decision problem k-DISTANCE EDGE COVER is NP-complete for planar bipartite graphs.

Corollary

k-DISTANCE EDGE COVER is NP-complete for planar graphs.

Corollary

k-DISTANCE EDGE COVER is NP-complete for bipartite graphs.

NP-completeness results

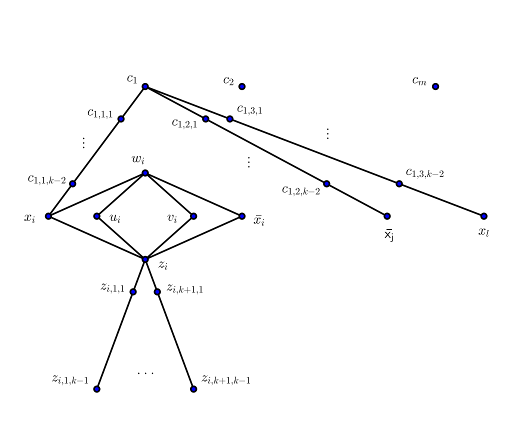


Figure : A transformation to a planar bipartite graph.

Nowhere-zero Flow and Modular Orientation

S.-H. Chen and G. J. Chang,

Nowhere-zero flows and modular orientations, manuscript.

Definitions

- Assume $G = (V, E)$ is an undirected graph and D is an orientation of $E(G)$.
- For a vertex $v \in V(G)$, let $E^+(v)$ ($E^-(v)$, resp) denote the set of directed edges with their tails (heads, resp) at the vertex v.

Definitions

- Suppose Γ is an abelian group.
- A Γ-**flow** of an undirected graph G is an ordered pair (D, f) where D is an orientation of $E(G)$ and $f : E(G) \longrightarrow \Gamma$ such that

$$\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0$$

for all $v \in V(G)$.

Definitions

- An **integer flow** (D, f) is a $\mathbb{Z}$-flow of G.
- An **integer k-flow** (or, simply **k-flow**) of G is an integer flow (D, f) such that $|f(e)| < k$ for all $e \in E(G)$.
- An Γ-flow (D, f) of G is **nowhere-zero** if $f(e) \neq 0$ for all $e \in E(G)$.

Nowhere-zero k-flows

- The concept of integer flow was introduced by Tutte (1949) as a refinement and a generalization of coloring problem of planar graphs.
- If G has an edge-cut, then it is impossible that G has a nowhere-zero k-flow for any $k \geq 2$, hence bridgeless is necessary.

Nowhere-zero k-flows

Proposition (Tutte, 1956)

A graph G has a nowhere-zero 2-flow if and only if the degree of each vertex is even.

Theorem (Seymour, 1981)

Every bridgeless graph has a nowhere-zero 6-flow.

Nowhere-zero k-flows

Conjecture

(5-flow Conjecture, Tutte, 1956) Every bridgeless graph admits a nowhere-zero 5-flow.

Conjecture

(4-flow Conjecture, Tutte, 1966) Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

Conjecture

(3-flow Conjecture, Steinberg, 1976) Every bridgeless graph containing no 3-edge-cut admits a nowhere 3-flow.

Modular k-orientation

- An orientation D of G is called a **modular k-orientation** if

$$d_D^+(v) \equiv d_D^-(v) \pmod{k}$$

  for each vertex $x \in V(G)$.

- The concept of modular orientation is introduced by Jaeger (1988).

Modular k-orientation

For k even, the situation is completely known.

Proposition

Let t be a positive integer. The following statements are equivalent.

1. G has a modular $(2t)$-orientation;
2. G is an Euler cycle.

Modular k-orientation

Proposition

The following statements are equivalent.

1. G has a modular $(2t+1)$-orientation;
2. G has a $\mathbb{Z}_{2t+1}$-flow (D_1, f_1) such that $f_1(e) = \pm 1$ for each edge $e \in E(G)$;
3. G has an integer flow (D_2, f_2) such that $f_2(e) \in \{\pm t, \pm(t+1)\}$ for each edge $e \in E(G)$;
4. G has an orientation D_3 such that
$$\frac{t}{t+1} \leq \frac{|[A,B]_{D_3}|}{|[B,A]_{D_3}|} \leq \frac{t+1}{t}$$
for every partition $\{A, B\}$ of $V(G)$.

Modular k-orientation

Conjecture (Jaeger, 1988)

Let $k \geq 3$ be an odd integer. Every $(2k-2)$-edge connected graph has a modular k-orientation.

- Jaeger's circular flow conjecture.
- It implies 3-flow, 5-flow conjecture.

Splitting-off lemma

Lemma (Splitting-off Lemma)

Suppose $G = (V + s, E)$ *is a graph such that* s *has even degree and* $\deg(s) \geq 2$. *Assume that*

$$\text{for each nonempty proper vertex subset } U \text{ of } V(G) : d(U) \geq k$$

where $k \geq 2$. *Then for every edge* $su \in E(G)$ *there exists an edge* $sv \in E(G)$ *such that in the graph* $G' = (V + s, E \setminus \{su, sv\} \cup \{uv\})$,

$$\text{for each nonempty proper vertex subset } U \text{ of } V(G) : d'(U) \geq k.$$

Splitting-off lemma

Theorem

Suppose $G = (V + s, E)$ *is a graph such that* $\deg(s) = k + 2$ *for some* $k \geq 2$*. Assume that*

for each nonempty proper vertex subset U *of* $V(G) : d(U) \geq k$

where $k \geq 2$*. Then there exists two edges* $su, sv \in E(G)$ *such that in the graph* $G' = (V + s, E \setminus \{su, sv\} \cup \{uv\})$,

for each nonempty proper vertex subset U *of* $V(G) : d'(U) \geq k$.

Splitting-off lemma

Corollary

Suppose $G = (V, E)$ is k-edge-connected and $\deg(s) \geq k + 2$ for some vertex $s \in V(G)$. Then there exists $u, v \in N(s)$ such that let $G' = (V + s, E \setminus \{su, sv\} \cup \{uv\})$, G' is also k-edge-connected.

Corollary

If G is the minimum counterexample to 3-flow conjecture, then G is 4-edge-connected and 5-regular.

Corollary

If G is the minimum counterexample to 5-flow conjecture, then G is 2-edge-connected and 3-regular.

Modified Splitting-off lemma

Lemma (Modified Splitting-off Lemma)

Suppose $G = (V + s, E)$ is a graph such that s has even degree and $\deg(s) \geq 2$. Assume that $t \notin N(s)$ and

$$\text{for each nonempty proper vertex subset } U \neq \{t\} : d(U) \geq k$$

where $k \geq 2$. Then for every edge $su \in E(G)$ there exists an edge $sv \in E(G)$ such that in the graph $G' = (V + s, E \setminus \{su, sv\} \cup \{uv\})$,

$$\text{for each nonempty proper vertex subset } U \neq \{t\} : d'(U) \geq k.$$

Modified Splitting-off lemma

Theorem

Suppose $G = (V + s + t, E)$ is a k-edge-connected graph.

(i) If $\deg(s) = 2l$ for some $l \in \mathbb{N}$, then there exists

$$G' = (V + t, E - \{su_i\}_{i=1}^{2l} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l})$$

where $u_1, u_2, \ldots, u_{2l} \in N(s)$ such that G' is k-edge-connected.

Modified Splitting-off lemma

Theorem (cont.)

(ii) If $st \in E(G)$, $\deg(s) = 2l + 1$ *and* $\deg(t) > k$, *then there exists*

$$G' = (V + t, E - \{st\} - \{su_i\}_{i=1}^{2l} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l})$$

where $u_1, u_2, \ldots, u_{2l} \in N(s) \setminus \{t\}$ *such that* G' *is* k*-edge-connected.*

(iii) If $st \in E(G)$, $\deg(s) = 2l + 1$ *and* $\deg(t) = 2m + 1$, *then there exists*

$$G' = (V, E - \{st\} - \{su_i\}_{i=1}^{2l} - \{tv_i\}_{i=1}^{2m} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l} \cup \{v_{2j-1}v_{2j}\}_{j=1}^{m})$$

where $u_1, u_2, \ldots, u_{2l} \in N(s) \setminus \{t\}$ *and* $v_1, v_2, \ldots, v_{2m} \in N(t) \setminus \{s\}$ *such that* G' *is* k*-edge-connected.*

Modified Splitting-off lemma

- The concept modular k-orientation can be generalized into **modular β-orientation**.
- A vertex mapping $\beta : V(G) \to \mathbb{Z}_k$ is called a $\mathbb{Z}_k$-**boundary** if $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{k}$. And an orientation D of G is called a **modular β-orientation** if, for every vertex $v \in V(G)$,

$$d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}.$$

Modified Splitting-off lemma

Theorem

Suppose $G = (V + s + t, E)$ is a graph.

(i) Let $G' = (V + s + t, E - st)$. If G' has a modular β-orientation for all β, then G has a modular β-orientation for all β.

Modified Splitting-off lemma

Theorem (cont.)

(ii) Assume $\deg(s) = 2l$ *for some* $l \in \mathbb{N}$. *Let*

$$G' = (V + t, E - \{su_i\}_{i=1}^{2l} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l})$$

where $N(s) = \{u_1, u_2, \ldots, u_{2l}\}$. *If* G' *has a modular* β*-orientation for all* β, *then* G *has a modular* β*-orientation with prescribed* $\beta(s) = 0$.

Modified Splitting-off lemma

Theorem (cont.)

(iii) Assume $st \in E(G)$, $\deg(s) = 2l + 1$ *and* $\deg(t) > k$. *Let*

$$G' = (V + t, E - \{st\} - \{su_i\}_{i=1}^{2l} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l})$$

where $N(s) = \{t, u_1, u_2, \ldots, u_{2l}\}$. *If* G' *has a modular* β*-orientation for all* β*, then* G *has a modular* β*-orientation for all* β *with prescribed* $\beta(s) \neq 0$.

Modified Splitting-off lemma

Theorem (cont.)

(iv) Assume $st \in E(G)$, $\deg(s) = 2l + 1$ *and* $\deg(t) = 2m + 1$*.*
Let

$$G' = (V, E - \{st\} - \{su_i\}_{i=1}^{2l} - \{tv_i\}_{i=1}^{2m} \cup \{u_{2j-1}u_{2j}\}_{j=1}^{l} \cup \{v_{2j-1}v_{2j}\}_{j=1}^{m})$$

where $N(s) = \{t, u_1, u_2, \ldots, u_{2l}\}$ *and* $N(t) = \{s, v_1, v_2, \ldots, v_{2m}\}$*.*
If G' *has a modular* β*-orientation for all* β*, then* G *has a modular* β*-orientation for all* β *with prescribed* $\beta(s) = -\beta(t) \neq 0$*.*

τ-function

- The idea originated from Thomassen (2012) who proved that every 8-edge-connected graph has a nowhere-zero 3-flow.
- Suppose β is a $\mathbb{Z}_k$-boundary, define the mapping $\tau : V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm k\}$ such that for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} \beta(x) & \pmod{k}, \\ \deg(x) & \pmod{2}. \end{cases}$$

τ-function

- Note that, if $\beta(x) = 0$ and $\deg(x)$ is odd, $\tau(x)$ has two possible values k or $-k$, we do not determine yet.
- Moreover, for other cases,

$$\tau(x) \equiv \begin{cases} \beta(x) & \text{if } \deg(x) - \beta(x) \text{ is even,} \\ \beta(x) - k & \text{if } \deg(x) - \beta(x) \text{ is odd.} \end{cases}$$

τ-function

- The mapping τ can be extended to any nonempty vertex subset A with respect to $\beta(A) \equiv \sum_{x \in A} \beta(x)$ and $d(A) = |[A, A^c]|$.
- The extended mapping $\tau : \mathcal{P}(V(G)) \to \{0, \pm 1, \pm 2, \ldots, \pm k\}$ is defined as follows, for each non-empty $A \subseteq V(G)$,

$$\tau(A) \equiv \begin{cases} \beta(A) & (\text{mod } k) \\ d(A) & (\text{mod } 2), \end{cases}$$

where $\mathcal{P}(V(G))$ is the power set of $V(G)$.

τ-function

Proposition

Let A be a vertex subset of G, and $a \in \mathbb{N}$. If $d(A) \geq 2a$, then

$$d(A) \geq (2a - k + 1) + |\tau(A)|.$$

Proposition

Let A, B, C be three disjoint vertex subsets of G with $A \cup B \cup C = V(G)$. Then

$$|\tau(A)| + |\tau(B)| \geq |\tau(C)|.$$

(p, q)-extendable

- Let $k \geq 3$ be an odd integer and $p(k), q(k)$ be two integer functions of k.
- In practical use, $p(k), q(k)$ are both even, hence we might assume that both $p(k)$ and $q(k)$ are even.
- G is $(p(k), q(k))$-**extendable** (or simply (p, q)-**extendable**) if the following statement holds:

(p, q)-extendable

For any Z_k-boundary β and $z_0 \in V(G)$,

$V_0 = \{v \in V(G) - z_0 \colon \tau(v) = 0\}$, and v_0 is a vertex of V_0 with $V_0 \neq \phi$. If

1. $d(z_0) \leq p(k) + |\tau(z_0)|$ and
2. $d(A) \geq q(k) + |\tau(A)|$ for any nonempty subset $A \subseteq V(G) - z_0$ with $|V(G) - A| > 1$ but $A \neq \{v_0\}$,

then any pre-orientation of $E(z_0)$ with $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{k}$ can be extended to a β-orientation of G, that is, $d^+(x) - d^-(x) \equiv \beta(x) \pmod{k}$ for every vertex $x \in V(G)$.

(p, q)-extendable

Theorem

If every graph is $(p(k), q(k))$-extendable, then every $(q(k) + k - 1)$-edge-connected graph has a modular β-orientation for every $\mathbb{Z}_k$-boundary β of G.

Theorem (Lovász *et al.*, 2013)

Every graph is $(2k - 2, 2k - 2)$-extendable.

Corollary (Lovász *et al.*, 2013)

Every $(3k - 3)$-edge-connected graph has a modular β-orientation for every $\mathbb{Z}_k$-boundary β of G.

(p, q)-extendable

Theorem

If every graph is $(p(k), q(k))$-extendable, then $p(k) \leq q(k)$.

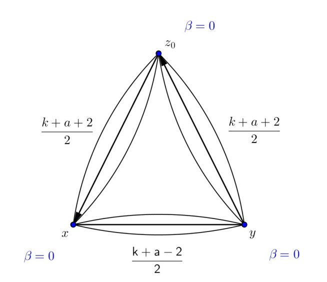


Figure : A non-$(k + a + 2, k + a)$-extendable graph.

(p, q)-extendable

Hence we focus on $p(k) = q(k)$, we say G is $q(k)$-**extendable** if G is $(q(k), q(k))$-extendable.

Theorem

If every graph is $q(k))$-extendable, then every $(q(k) + k - 1)$-edge-connected graph has a modular β-orientation for every $\mathbb{Z}_k$-boundary β of G.

(p, q)-extendable

Conjecture

Every graph is $(k+1)$-extendable.

Theorem (Lovász *et al.*, 2013)

Every graph is $(2k-2)$-extendable.

(p, q)-extendable

Theorem

Let $k \geq 5$ be an odd integer. If every graph is $q(k)$-extendable, then $q(k) \geq 2\lceil \frac{3k+1}{4} \rceil$.

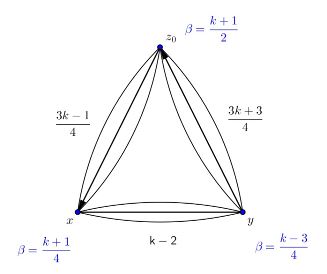


Figure : A non-$(\frac{3k-5}{2})$-extendable graph.

(F, Γ)-flow

- Let F is a subset of additive abelian group Γ.
- A flow is (F, Γ)-**flow** if all flow values are elements in F.
- We define $f(F, \Gamma)$ to be the **minimum edge-connectivity of** (F, Γ)-**flow** as the smallest natural number such that every $f(F, \Gamma)$-edge-connected (finite) graph G has a (F, Γ)-flow.

(F, Γ)-flow

- This concept is recently introduced by Thomassen (2014).

Theorem (Thomassen, 2014)

Let $a_1, a_2, \ldots, a_{2p}, b_1, b_2, \ldots, b_{2q+1}$ be elements in an additive abelian group Γ such that

$$a_1 + a_2 + \cdots + a_{2p} = b_1 + b_2 + \cdots + b_{2q+1}.$$

Put $k = 2p + 2q + 1$. If G is a graph with edge-connectivity at least $3k - 1$, then G has a flow whose flow values are in $\{a_1, a_2, \ldots, a_{2p}, b_1, b_2, \ldots, b_{2q+1}\}$.

(F, Γ)-flow

- If $|F| = 1$, the element of F must have finite odd order.
- In fact, an $(\{1\}, \mathbb{Z}_k)$-flow is just a modular k-orientation.

Theorem (Thomassen, 2014)

Let $k \geq 3$ be an odd integer. Then

$$2k - 2 \leq f(\{1\}, \mathbb{Z}_k) \leq 3k - 3.$$

(F, Γ)-flow

Theorem (Thomassen, 2014)

Let Γ be an abelian group and $F = \Gamma - \{0\}$.

1. *If $|\Gamma| \geq 6$, then $f(F, \Gamma) = 2$.*
2. *If $|\Gamma| = 5$, then $f(F, \Gamma) = 2$ or $f(F, \Gamma) = 4$.*
3. *If $|\Gamma| = 4$, then $f(F, \Gamma) = 4$.*
4. *If $|\Gamma| = 3$, then $f(F, \Gamma) = 4$ or $f(F, \Gamma) = 6$.*

(F, Γ)-flow

Theorem

Let $k \geq 3$ be an odd integer. Then $2k - 2 < f^*(\{1\}, \mathbb{Z}_k) \leq 3k - 3$.

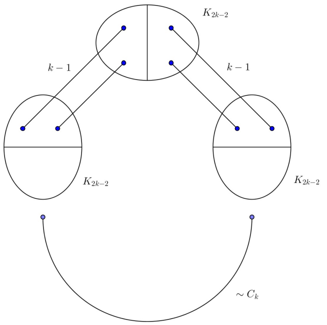


Figure : A $2k - 2$-edge-connected graph without modular β-orientation.

5-flow conjecture

Proposition (Jaeger, 1988)

If $f(\{1\}, \mathbb{Z}_5) \leq 9$, then every bridgeless graph has a nowhere-zero 5-flow. Note that $f^*(\{1\}, \mathbb{Z}_5) \leq 9$ implies $f(\{1\}, \mathbb{Z}_5) \leq 9$.

Corollary

$8 \leq f(\{1\}, \mathbb{Z}_5) \leq 12$ *and* $9 \leq f^*(\{1\}, \mathbb{Z}_5) \leq 12$.

Circular-flow

Proposition

For every positive integer t, a graph G has a nowhere-zero circular $(2 + \frac{1}{t})$-flow if and only if G has a modular $(2t + 1)$-orientation.

Theorem (Lovász *et al.*, 2013)

For every natural number t, every $6t$-edge-connected graph has a nowhere-zero circular $(2 + \frac{1}{t})$-flow.

Corollary

For every natural number t, every $f(\{1\}, \mathbb{Z}_{2t+1})$-edge-connected graph has a nowhere-zero circular $(2 + \frac{1}{t})$-flow.

Relaxation Procedure

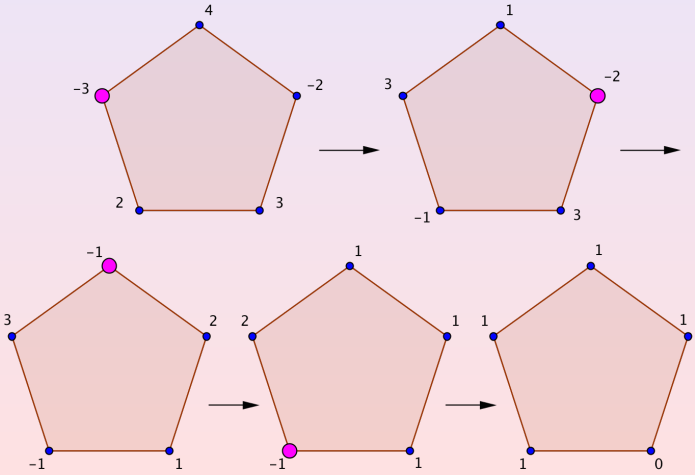
S.-H. Chen and G. J. Chang,

Relaxation procedures on graphs, submitted.

The pentagon game

- An interesting game was proposed at International Mathematical Olympiad in 1986.
- **The pentagon game:** Five integers with positive sum are assigned to the vertices of a pentagon. If there is at least one negative number, the player can choose one of them, then reverse the sign and add it to its two neighbors. The game terminates when all numbers are nonnegative. Prove that the pentagon game always terminates.

The pentagon game



Definitions

- The game was generalized by Wegert and Reiher (2009) from a pentagon to connected graphs.
- Suppose G is a connected with $V(G) = \{v_1, v_2, \ldots, v_n\}$.
- An n-tuple $X = (x_1, x_2, \ldots, x_n)$ of real numbers is called a **configuration** of G if each vertex v_i in G is assigned with the label x_i, and suppose the sum $s = \sum_{i=1}^{n} x_i$ is positive.

Definitions

- If there is a negative label x_i, then a **legal relaxation** $R^{(i)}$ for X is defined as the operation which transform X into $X' = XR^{(i)} = (x_1', x_2', \ldots, x_n')$ obtained from replacing x_i by $-x_i > 0$ and add $2x_i/d_i$ to each of the d_i neighbors of v_i. That is, $x_i' = -x_i$, $x_j' = x_j + \frac{2}{d_i}x_i$ for each v_j adjacent to v_i, and $x_k' = x_k$ for all other k.
- Note that the sum $s' = \sum_{i=1}^{n} x_i' = \sum_{i=1}^{n} x_i = s$ is unchanged and the connectedness of G can be omitted if we assume that $s > 0$ holds in every component of the graph.

Relaxation Procedure

- A **relaxation procedure** for X of G is a sequence of configurations $X = X_0, X_1, X_2, \ldots$ and a sequence of relaxations $R^{(k_1)}, R^{(k_2)}, \ldots$ such that $X_i = X_{i-1}R^{(k_i)}$ for $i \geq 1$.
- We say that a relaxation procedure **terminates** if all the elements of X_t are nonnegative for some t, that is, there is no legal relaxation for X_t.

Relaxation Procedure

Theorem (Wegert and Reiher, 2009)

If G is a connected graph and X is an n-tuple of real numbers with positive sum, then a relaxation procedure for X of G always terminates.

- $X_t = XR^{(k_1)}R^{(k_2)} \dots R^{(k_t)}$ is called a **final configuration** of the **initial configuration** X if all its elements are nonnegative, and t is called **the number of steps** of the relaxation procedure.

Independence

- Note that t and X_t may depend on the relaxation procedure.
- Our goal is to characterize connected graphs for which the final configurations and/or the numbers of steps are unique for any initial configuration.

Non-uniqueness

Lemma

If v_i and v_j are two adjacent vertices with $\deg(v_i)\deg(v_j) \neq 1, 2, 4$ in a connected graph G, then there exists an initial configuration X and two relaxation procedures which both the final configurations and the numbers of steps are different.

Non-uniqueness

Sketch of proof:

- If $\deg(v_i) = 1$ and $\deg(v_j) = 3$, then consider the initial configuration $X = (x_1, x_2, \ldots, x_n)$ with $x_i = -15, x_j = -9$, and for $k \neq i, j$, x_k is large enough to keep them positive during the procedure.
- Then observe the changing on (x_i, x_j):

$$(-15, -9) \xrightarrow{R^{(i)}} (15, -39) \xrightarrow{R^{(j)}} (-11, 39) \xrightarrow{R^{(i)}} (11, 17)$$

but

$$(-15, -9) \xrightarrow{R^{(j)}} (-21, 9) \xrightarrow{R^{(i)}} (21, -33) \xrightarrow{R^{(j)}} (-1, 33) \xrightarrow{R^{(i1)}} (1, 31).$$

Non-uniqueness

Sketch of proof (cont.):

- If $\deg(v_i) = p, \deg(v_j) = q$ with $pq \geq 5$, then consider $X = (x_1, x_2, \ldots, x_n)$ with $x_i = -2p^2q^2, x_j = -pq^2$, and for $k \neq i, j$, x_k is large enough to keep them positive during the procedure.
- Then observe the changing on (x_i, x_j):

$$(-2p^2q^2, -pq^2) \xrightarrow{R^{(i)}} (2p^2q^2, -5pq^2) \xrightarrow{R^{(j)}} (2p^2q^2 - 10pq, 5pq^2)$$

$$\begin{aligned}(-2p^2q^2, -pq^2) &\xrightarrow{R^{(j)}} (-2p^2q^2 - 2pq, pq^2)\\ &\xrightarrow{R^{(i)}} (2p^2q^2 + 2pq, -3pq^2 - 4q)\\ &\xrightarrow{R^{(j)}} (2p^2q^2 - 4pq - 8, 3pq^2 + 4q).\end{aligned}$$

Uniqueness

- Notice that according to this lemma, all connected graphs, except cycles C_n, paths P_n and $K_{1,4}$, have more than one final configuration and more than one number of steps for some initial configuration.

Theorem (Alon, Krasikov and Peres, 1989)

If $G = C_n$ is the n-cycle ($n \geq 3$) with an initial configuration X, then the number of steps and the final configuration of any relaxation procedure are independent to the relaxation procedures.

Uniqueness

Lemma

Suppose G is a graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$ with an initial configuration $X = (x_1, x_2, \ldots, x_n)$ with $x_i, x_j < 0$.

1. *If v_i is not adjacent to v_j, then $XR^{(i)}R^{(j)} = XR^{(j)}R^{(i)}$.*
2. *If v_i is adjacent to v_j and $\deg(v_i)\deg(v_j) = 4$, then $XR^{(i)}R^{(j)}R^{(i)} = XR^{(j)}R^{(i)}R^{(j)}$.*
3. *If v_i is adjacent to v_j and $\deg(v_i)\deg(v_j) = 2$, then $XR^{(i)}R^{(j)}R^{(i)}R^{(j)} = XR^{(j)}R^{(i)}R^{(j)}R^{(i)}$.*

Uniqueness

Theorem

If $G = C_n$ $(n \geq 3)$, P_n $(n \geq 1)$ or $K_{1,4}$ with an initial configuration X, then the number of steps and the final configuration of any relaxation procedure are independent to the relaxation procedures.

Uniqueness

Corollary

The following statements are equivalent for any graph.

1. *Each component of the graph is* C_n, P_n *or* $K_{1,4}$.
2. *For any initial configuration* X *with positive sum in each component, the number of steps is the same whatever relaxation procedures take.*
3. *For any initial configuration* X *with positive sum in each component, the final configuration is independent to the relaxation procedure.*

Thanks for your attention.